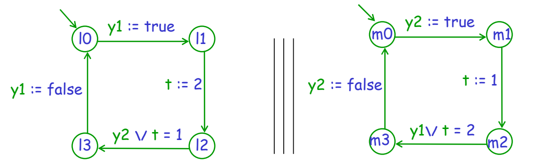
Lecture 1
Branching time logics and CTL

Peterson-Fischer: Possible Specifications

Variables: y1, y2: boolean, t: {1,2}
Initially y1 = y2 = false, t = 1

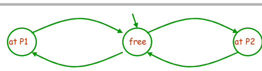


Mutual Exclusion: the two processes never simultaneously reach l3,m3

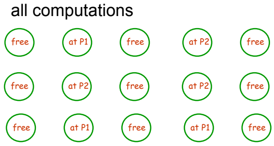
Absence of Starvation: If the left process is at l1, it will later reach l3

Bounded Overtaking: If the left process is at l1, the other process will reach m3 at most once (twice?) before the left process reaches l3

Two Classes of Formalizations

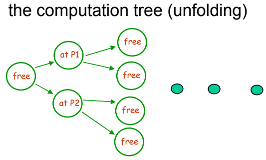


Linear Time Properties:
Specify properties that hold for all computations



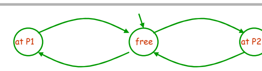
Free in every 2nd state ;
at P1 always followed by free

Branching Time Properties:
Specify properties that hold for the computation tree (unfolding)



always possible to reach free ;
always possible to reach at P1 ;
free is always eventually reached

Two Classes of Formalizations

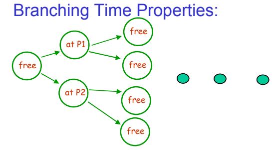


Linear Time Properties:

Specification says:
In all computations,
Free in every 2nd state ;

In all computations,
at P1 always followed by free

Branching Time Properties:



In all computations,
always
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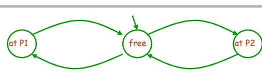

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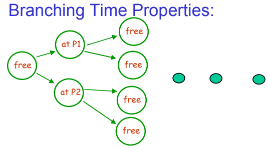

In all computations (paths),
always
in some computation (path)
possible to reach free

Structure of CTL Formulas

Operators in branching time logic typically have two components:
<in all/some (future) path(s)> <temporal property>

CTL syntax:
A/E G/F/U φ

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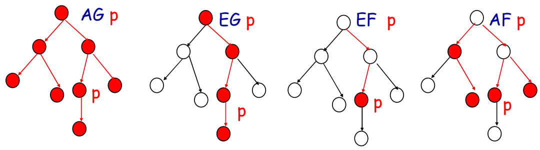
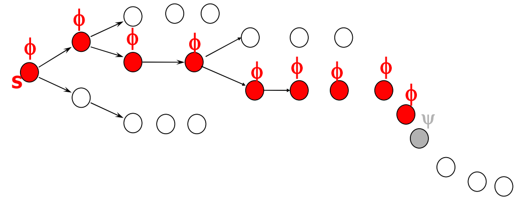
Invariant **Potentially globally** **Possible** **Inevitable**

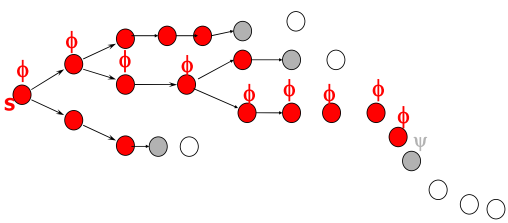


E[ϕ U ψ]



E[ϕ U ψ] is valid in **s** if <u>some path</u> from **s** satisfies the above

A[ϕ U ψ]



A[ϕ U ψ] is valid in **s** if <u>all pathes</u> from **s** satisfy the above

Computation Tree Logic, CTL

Clarke & Emerson 1980

Syntax

ϕ **::= P | $\neg\,\phi$ | $\phi \vee \phi$ | EX ϕ | E[ϕ U ϕ] | A[ϕ U ϕ]**

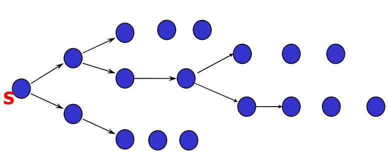
where **P** $\in$ AP (atomic propositions)

- EX (pronounced "for some path next")
- E (pronounced "for some path")
- A (pronounced "for all paths") and
- U (pronounced "until").

Computations vs. States

Linear time formula satisfied by state in certain computation
Branching time formula satisfied by state in the computation tree.
Meaning of nested formula different

$\Diamond\Box$ p



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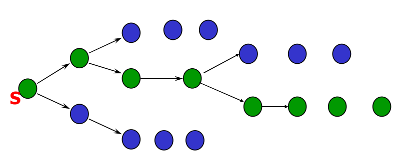

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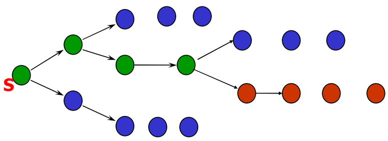

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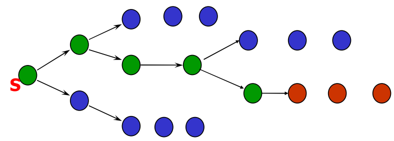

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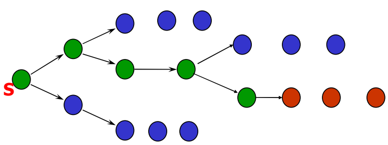

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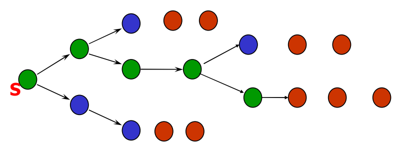

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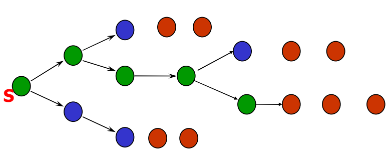
In branching time AF (A G p)
But then, we cannot let each computation "decide" when to start G p



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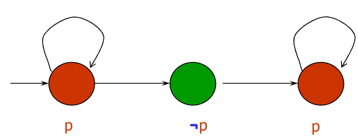

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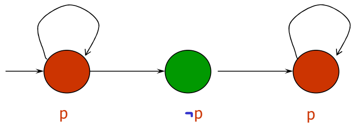
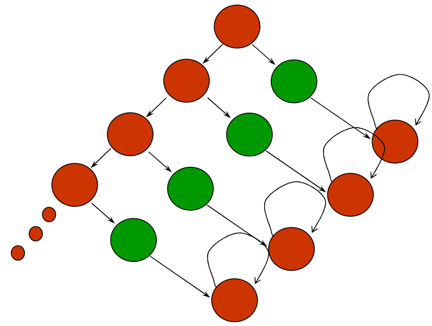


This Transition system satisfies ◇□ p
But not AF (A G p)

Why?



Formal Semantics of CTL

$$
\begin{array}{ll}
s \models p & \text{iff } p \in \mathit{Label}(s) \\
s \models \neg\phi & \text{iff } \neg(s \models \phi) \\
s \models \phi \vee \psi & \text{iff } (s \models \phi) \vee (s \models \psi) \\
s \models \mathsf{EX}\,\phi & \text{iff } \exists \sigma \in P_{\mathcal{M}}(s).\, \sigma[1] \models \phi \\
s \models \mathsf{E}\,[\phi\,\mathsf{U}\,\psi] & \text{iff } \exists \sigma \in P_{\mathcal{M}}(s).(\exists j \geqslant 0.\, \sigma[j] \models \psi \;\wedge\; (\forall 0 \leqslant k < j.\, \sigma[k] \models \phi)) \\
s \models \mathsf{A}\,[\phi\,\mathsf{U}\,\psi] & \text{iff } \forall \sigma \in P_{\mathcal{M}}(s).(\exists j \geqslant 0.\, \sigma[j] \models \psi \;\wedge\; (\forall 0 \leqslant k < j.\, \sigma[k] \models \phi)).
\end{array}
$$

Where $P_M(s)$ denotes the set of paths starting from s
and $\sigma[i]$ denotes i'th element of σ

Theorem

All operators are derivable from

- EX *f*
- EG *f*
- E[*f* U *g*]

and boolean connectives

$$A[f\,\mathsf{U}\,g] \equiv \neg E[\neg g\,\mathsf{U}(\neg f \wedge \neg g)] \wedge \neg EG\neg g$$

CTL Model-Checking Algorithms

Labeling Methods [Clarke et al 81]

- Sat(ϕ) = all states where ϕ is true
- Compute Sat(ϕ) recursively as follows:
  - For each sub-formula ϕ_i of ϕ, compute Sat(ϕ_i)
  - This is easier: e.g. Sat(P) = {s | P$\in$Label(s)}
- Compose Sat(ϕ_i) to get Sat(ϕ)

Labeling Methods [Clarke et al 81]

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- Compose $Sat(\phi_i)$ to get $Sat(\phi)$

This is a **GLOBAL** model checking algorithm:

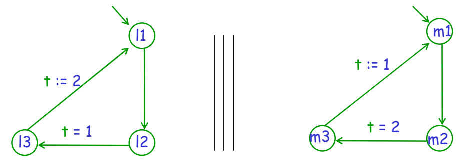
Compute all states that satisfy ϕ

Compare with **LOCAL** model checking algorithm:

Check whether some (e.g., initial) state satisfies ϕ

Simpler mutex

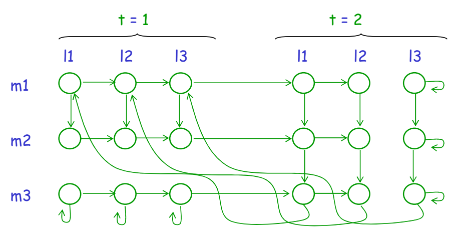
Variables: t: {1,2}
Initially t = 1



Mutual Exclusion: AG ¬(at l2 /\ at m3)

Absence of Starvation: AG (at l2 -> AF at l3)
Bounded Overtaking: AG(at l2 -> A[¬at m3 U A[at m3 U A[¬at m3 U at l3]]])

Simpler state space



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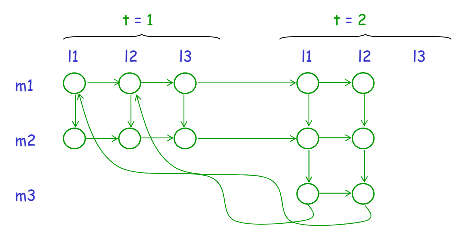

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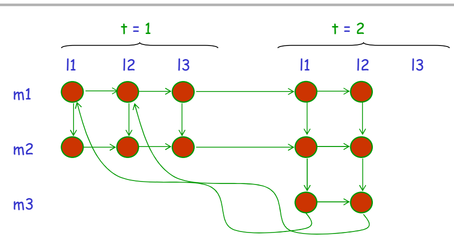

Absence of Starvation: AG (at l2 -> AF at l3)

function $Sat(\phi : Formula)$: **set of** $State$;
(* precondition: true *)
begin
  if $\phi = \text{true} \longrightarrow$ **return** S
  [] $\phi = \text{false} \longrightarrow$ **return** $\varnothing$
  [] $\phi \in AP \longrightarrow$ **return** $\{ s \mid \phi \in Label(s) \}$
  [] $\phi = \neg \phi_1 \longrightarrow$ **return** $S - Sat(\phi_1)$
  [] $\phi = \phi_1 \lor \phi_2 \longrightarrow$ **return** $(Sat(\phi_1) \cup Sat(\phi_2))$
  [] $\phi = \mathsf{EX}\,\phi_1 \longrightarrow$ **return** $\{ s \in S \mid (s, s') \in R \ \land \ s' \in Sat(\phi_1) \}$
  [] $\phi = \mathsf{E}\,[\phi_1 \,\mathsf{U}\, \phi_2] \longrightarrow$ **return** $Sat_{EU}(\phi_1, \phi_2)$
  [] $\phi = \mathsf{A}\,[\phi_1 \,\mathsf{U}\, \phi_2] \longrightarrow$ **return** $Sat_{AU}(\phi_1, \phi_2)$
  fi
(* postcondition: $Sat(\phi) = \{ s \mid \mathcal{M}, s \models \phi \}$ *)
end

How to Compute Sat(E[ϕ U ψ])

function $Sat_{EU}(\phi, \psi : Formula)$: set of State;
(* precondition: true *)
begin var Q, Q' : set of State;
$Q, Q' := Sat(\psi), \varnothing$;
do $Q \neq Q' \longrightarrow$
$Q' := Q$;
$Q := Q \cup (\{s \mid \exists s' \in Q.(s,s') \in R\} \cap Sat(\phi))$
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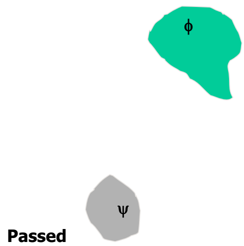
Table 3.4: Labelling procedure for E[ϕ U ψ]



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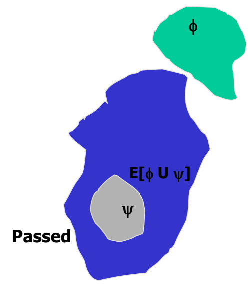

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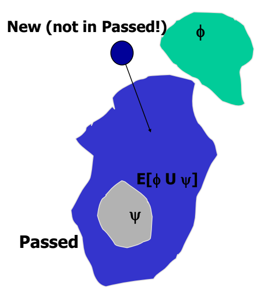

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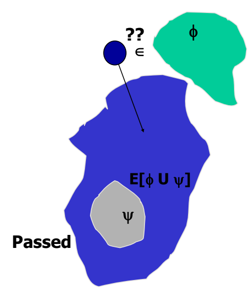

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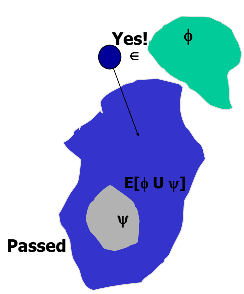

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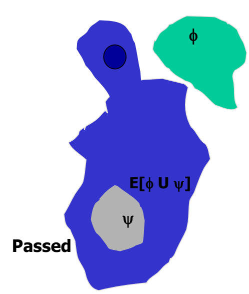

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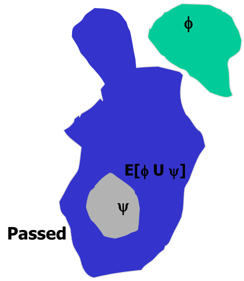

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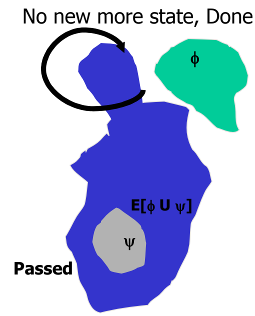

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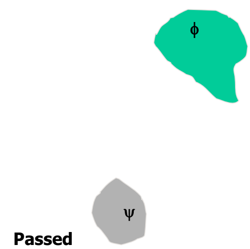

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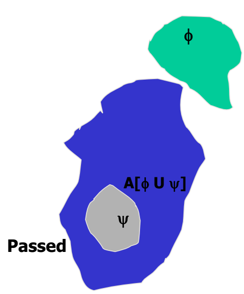

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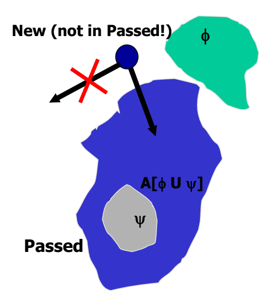

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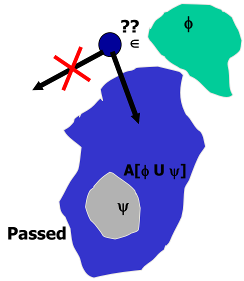

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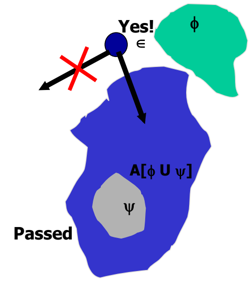
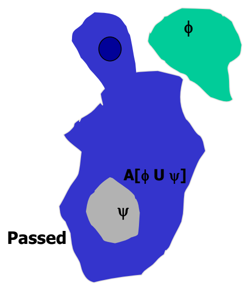
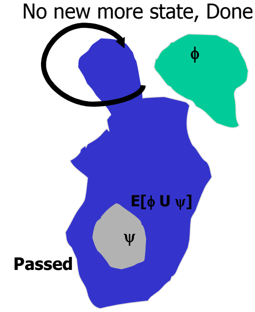


How to Compute Sat(A[ϕ U ψ])



How to Compute Sat(A[ϕ U ψ])

No new more state, Done!



function $Sat_{AU}\,(\phi, \psi : Formula)$: **set of** $State$;
(* precondition: true *)
begin var Q, Q' : **set of** $State$;
$\quad Q, Q' := Sat(\psi), \varnothing$;
$\quad$ **do** $Q \neq Q' \longrightarrow$
$\quad\quad Q' := Q$;
$\quad\quad Q := Q \cup (\{s \mid \forall s'.(s,s') \in R \Rightarrow s' \in Q\} \cap Sat(\phi))$
$\quad$ **od**;
$\quad$ **return** Q
(* postcondition: $Sat_{AU}(\phi, \psi) = \{ s \in S \mid \mathcal{M}, s \models \mathsf{A}[\phi \,\mathsf{U}\, \psi] \}$ *)
end

Table 3.5: Labelling procedure for $\mathsf{A}[\phi \,\mathsf{U}\, \psi]$

More efficient algorithm for A[φ U ψ]

The preceding algorithm has quadratic complexity.
We can do better by excluding states that satisfy

E[¬ψ W (¬φ /\ ¬ψ)]

1. Include states that satisfy ψ
2. Exclude states that satisfy ¬φ /\ ¬ψ
3. Exclude states that satisfy φ /\ ¬ψ and either
   1. can reach a state that satisfies ¬ φ /\ ¬ψ
   2. or are blocked
   3. or are in a non-trivial (with at least 2 states) strongly connected component in states that satisfy ¬ φ /\ ¬ψ

Fixpoint Characterizations (SMV)

$$\text{EF}\,p \;\equiv\; p \vee \text{EXEF}\,p$$

Let A be the set of states satisfying EF p then

$$\text{A} \;\equiv\; p \vee \text{EXA}$$

in fact A is the smallest one of sets satisfying the equations (the least fixpoint)

Fixed points of monotonic functions

- Let τ be a function $S \rightarrow S$
- Say τ is *monotonic* when

$$x \subseteq y \text{ implies } \tau(x) \subseteq \tau(y)$$

- Fixed point of τ is y such that

$$\tau(y) = y$$

- If τ monotonic, then it has
  - least fixed point $\mu y.\ \tau(y)$
  - greatest fixed point $\nu y.\ \tau(y)$

Iteratively computing fixed points

- Suppose S is finite
  - The least fixed point $\mu y.\ \tau(y)$ is the limit of

$$\text{false} \subseteq \tau(\text{false}) \subseteq \tau(\tau(\text{false})) \subseteq \Lambda$$

  - The greatest fixed point $\nu y.\ \tau(y)$ is the limit of

$$\text{true} \supseteq \tau(\text{true}) \supseteq \tau(\tau(\text{true})) \supseteq \Lambda$$

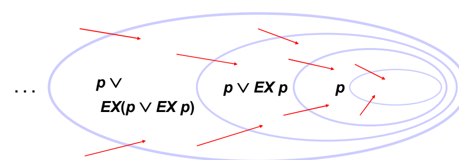
Note, since S is finite, convergence is finite

Example: *EF p*

- *EF p* is characterized by

$$EF\ p = \mu y.\,(p \vee EX\ y)$$

- Thus, it is the limit of the increasing series...

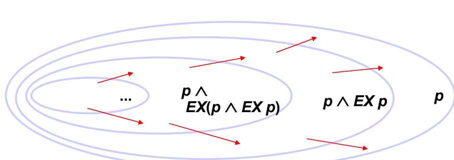


Example: *EG p*

- *EG p* is characterized by

$$EG\ p = \nu y.\ (p \wedge EX\ y)$$

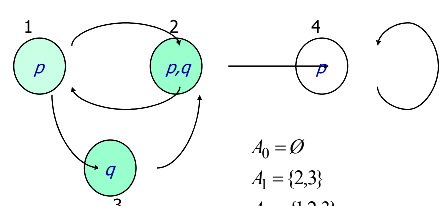
- Thus, it is the limit of the decreasing series...



Example, continued

EF *q*

$$EF\ q = \mu y.\,(q \vee EX\ y)$$



$$A_0 = \varnothing$$
$$A_1 = \{2,3\}$$
$$A_2 = \{1,2,3\}$$
$$A_3 = \{1,2,3\}$$

Remaining operators

$$\begin{aligned} AF\,p &= \mu y.(p \vee AX\,y) \\ AG\,p &= \nu y.(p \wedge AX\,y) \\ E(pUq) &= \mu y.(q \vee (p \wedge EX\,y)) \\ A(pUq) &= \mu y.(q \vee (p \wedge AX\,y)) \end{aligned}$$

Complexity

The worst-case time complexity of checking whether system-model sys satisfies the CTL*-formula* ϕ *is* $\mathcal{O}(|S_{sys}|^2 \times |\phi|)$

However S_{sys} may be EXPONENTIAL in number of parallel components!
--
FIXPOINT COMPUTATIONS may be carried out using
ROBDD's
(Reduced Ordered Binary Decision Diagrams)
Bryant, 86

Course Material

You will **need**

- Lecture Handouts
- A few papers (will be distributed)
- SPIN documentation

Reference texts: You can from

- SPIN MODEL CHECKER Primer and Reference Manual by Gerard Holzmann, good textbook for the course.
- *Design and Validation of Computer Protocols*, G.J. Holzmann, Prentice Hall 1991, older book, can be downloaded from the net.
- Old notes prepared by me some years ago.

I will borrow a few slides by

- G. Holzmann
- J.-P. Katoen
- T. Ruys

You can inspect these slides yourself (see the course page).

A CTL-model is a Kripke Structure (=transition systems with labeled nodes)

M = <S, E, Label> where

- S is a non-empty set of states
- $E \subseteq S \times S$ is a transition relation
- Label: $S \rightarrow 2^{AP}$ is a labeling function