Problem 1. Determine all composite integers \( n > 1 \) that satisfy the following property: if \( d_1, d_2, \ldots, d_k \) are all the positive divisors of \( n \) with \( 1 = d_1 < d_2 < \cdots < d_k = n \), then \( d_i \) divides \( d_{i+1} + d_{i+2} \) for every \( 1 \leq i \leq k - 2 \).

Problem 2. Let \( ABC \) be an acute-angled triangle with \( AB < AC \). Let \( \Omega \) be the circumcircle of \( ABC \). Let \( S \) be the midpoint of the arc \( CB \) of \( \Omega \) containing \( A \). The perpendicular from \( A \) to \( BC \) meets \( BS \) at \( D \) and meets \( \Omega \) again at \( E \neq A \). The line through \( D \) parallel to \( BC \) meets line \( BE \) at \( L \). Denote the circumcircle of triangle \( BDL \) by \( \omega \). Let \( \omega \) meet \( \Omega \) again at \( P \neq B \). Prove that the line tangent to \( \omega \) at \( P \) meets line \( BS \) on the internal angle bisector of \( \angle BAC \).

Problem 3. For each integer \( k \geq 2 \), determine all infinite sequences of positive integers \( a_1, a_2, \ldots \) for which there exists a polynomial \( P \) of the form \( P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1x + c_0 \), where \( c_0, c_1, \ldots, c_{k-1} \) are non-negative integers, such that

\[
P(a_n) = a_{n+1}a_{n+2} \cdots a_{n+k}
\]

for every integer \( n \geq 1 \).
Problem 4. Let \( x_1, x_2, \ldots, x_{2023} \) be pairwise different positive real numbers such that

\[
a_n = \sqrt{\left( x_1 + x_2 + \cdots + x_n \right) \left( \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right)}
\]

is an integer for every \( n = 1, 2, \ldots, 2023 \). Prove that \( a_{2023} \geq 3034 \).

Problem 5. Let \( n \) be a positive integer. A *Japanese triangle* consists of \( 1 + 2 + \cdots + n \) circles arranged in an equilateral triangular shape such that for each \( i = 1, 2, \ldots, n \), the \( i \)th row contains exactly \( i \) circles, exactly one of which is coloured red. A *ninja path* in a Japanese triangle is a sequence of \( n \) circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with \( n = 6 \), along with a ninja path in that triangle containing two red circles.

![Japanese Triangle](image)

In terms of \( n \), find the greatest \( k \) such that in each Japanese triangle there is a ninja path containing at least \( k \) red circles.

Problem 6. Let \( ABC \) be an equilateral triangle. Let \( A_1, B_1, C_1 \) be interior points of \( ABC \) such that \( BA_1 = A_1C, CB_1 = B_1A, AC_1 = C_1B \), and

\[
\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.
\]

Let \( BC_1 \) and \( CB_1 \) meet at \( A_2 \), let \( CA_1 \) and \( AC_1 \) meet at \( B_2 \), and let \( AB_1 \) and \( BA_1 \) meet at \( C_2 \). Prove that if triangle \( A_1B_1C_1 \) is scalene, then the three circumcircles of triangles \( AA_1A_2, BB_1B_2 \) and \( CC_1C_2 \) all pass through two common points.

(Note: a scalene triangle is one where no two sides have equal length.)