THE LARGEST RESPECTFUL FUNCTION

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ABSTRACT. Respectful functions were introduced by Sangiorgi as a compositional tool to formulate short and clear bisimulation proofs. Usually, the larger the respectful function, the easier the bisimulation proof. In particular the largest respectful function, defined as the pointwise union of all respectful functions, has been shown to be very useful. We here provide an explicit and constructive characterization of it.

1. INTRODUCTION

1.1. **Bisimulation and up-to techniques.** The well known method of bisimilarity for defining behavioural equivalence on labelled transition systems works as follows. A symmetric binary relation R is a *bisimulation* if whenever $p \; R \; q$ and $p \xrightarrow{a} p'$ then $q \xrightarrow{a} q'$ and $p' \; R \; q'$. In other words, whatever p can do can be mimicked by q such that the derivatives are still related. The idea is usually attributed to Park [Par81] although similar notions in logics and non well founded sets were present earlier, and it was popularized by Milner in his subsequent papers and book [Mil89] on communicating systems. Bisimulations are closed under union and therefore the union of them all is the largest bisimulation, written $\dot\sim$ and called *bisimilarity*. The main point is that if $p \dot\sim q$ then p and q can mimic each other indefinitely, and are thus inseparable for an observer who can only detect the labels of the transitions.

In order to establish $p \dot\sim q$ we must find a bisimulation relation containing the pair (p, q). On the one hand we would like this relation to be as small as possible, since for every pair in it we must check that all transitions can be mimicked. On the other hand we would also want it as large as possible in order to facilitate the proof of $p' \; R \; q'$, that the derivatives are related. These apparently conflicting interests were noted already by Milner [Mil89] who suggested a remedy: instead of requiring $p' \; R \; q'$ in the consequent it suffices to require $p' \dot\sim \circ R \circ \dot\sim q'$. This means that before establishing membership in R we are allowed to replace the derivatives with already known bisimilar ones. In many cases this makes the proofs significantly easier and clearer. Milner dubbed the technique *bisimulation up-to* $\dot\sim$ and it quickly caught on. Variants of it were used in many other process algebras, and other up-to techniques such as up-to one-hole contexts turned out to be useful.

The first to establish a systematic theory of up-to techniques was Sangiorgi [San94, San98], defining several important notions. One is a *progress* relation between binary relations. Briefly put the relation R progresses to the relation S, written $R \rightarrowtail S$, if whenever

$p \; R \; q$ and $p \stackrel{a}{\longrightarrow} p'$ then $q \stackrel{a}{\longrightarrow} q'$ and $p' \; S \; q'$, and vice versa. A bisimulation is thus a relation that progresses to itself. Milner's original up-to technique uses a relation R progressing to $\dot{\sim} \circ R \circ \dot{\sim}$. In general an up-to technique uses a function $\mathcal{F}$ on relations such that R progresses to $\mathcal{F}(R)$, i.e., if $p \; R \; q$ and $p \stackrel{a}{\longrightarrow} p'$ then $q \stackrel{a}{\longrightarrow} q'$ and $p' \; \mathcal{F}(R) \; q'$. The question then is which such $\mathcal{F}$ are *sound* in the sense that they guarantee that R is included in bisimilarity, and how they can be constructed in a systematic way.

1.2. **Respectful Functions.** Sangiorgi showed that although the sound functions in general are not closed under composition, a subset of them, called the *respectful* functions, are closed under composition, (pointwise) union and iteration. This means that complicated respectful functions can be constructed in a modular way from simple components, which clarifies and shortens bisimulation proofs: the effect is that up-to techniques based on respectful functions can be freely combined. The respectfulness criterion is that if $R \subseteq S$ and $R \rightarrowtail S$ then $\mathcal{F}(R) \subseteq \mathcal{F}(S)$ and $\mathcal{F}(R) \rightarrowtail \mathcal{F}(S)$. In other words, if a relation progresses to a superset then the same must hold when $\mathcal{F}$ is applied. Similar notions have lent structure to proofs in more advanced settings such as the pi-calculus [Hir97] and psi-calculi [PP16]. The idea has also been recast in a more general form by Pous [Pou07], where the corresponding notion is of so called *compatible* functions, though we shall here stick with Sangiorgi's original definitions and notations.

   In a typical bisimulation proof one begins with processes p and q to be proved bisimilar, and then defines a relation R containing them such that R progresses to $\mathcal{F}(R)$ for some respectful $\mathcal{F}$. It then follows that $R \subseteq \dot{\sim}$ and the proof is concluded. It does not matter which $\mathcal{F}$ is used as long as it is respectful. Often, the larger $\mathcal{F}(R)$ the easier the proof. Therefore a viable strategy is to simply choose $\mathcal{F}(R)$ as the largest of all possible images of R for a respectful function. This is well defined since the respectful functions are closed under arbitrary pointwise union; we can simply define the largest respectful function as the pointwise union of all respectful functions. In other words a relation R is mapped to the union of all $\mathcal{F}(R)$ for all respectful $\mathcal{F}$.

   To our knowledge the first explicit mention of something similar to this is in Hur et al. [HNDV13] where the largest respectful function is denoted by †. It is remarked that † is useful in coinductive proofs even though it lacks a constructive definition. The authors write "... the greatest respectful up-to function is so powerful, we see no point in ever stating a proof component's contribution involving a different respectful up-to function." In recent work [Pou16] Pous presents a similar idea, that of the largest compatible function, called the *companion*, and demonstrates its usefulness in a variety of settings. In a sense it is remarkable that although the notion of respectfulness has been around for more than 20 years, the largest and therefore most general respectful function has not been investigated and given an independent characterization.

   The contribution of this short note is to give an explicit characterization of the largest respectful function. It is easy to see that if $R \subseteq \dot{\sim}$ then it must map R to $\dot{\sim}$, but for other R the situation has been less clear. Our characterization uses Milner's stratification of bisimilarity [Mil89] $\sim_\alpha$ for any ordinal α. In brief our result is that the largest respectful function maps R to $\sim_\alpha$ where α is the largest ordinal such that $R \subseteq \sim_\alpha$. The following section provides detailed definitions and proof of this result. In the final section we briefly mention a generalisation to complete lattices and demonstrate the connection with the compatible functions by Pous.

2. Result

We assume a labeled transition system $(\mathbf{S}, \Delta, \longrightarrow)$ where $\mathbf{S}$ is a set of states, Δ is a set of labels, and $\longrightarrow \subseteq \mathbf{S} \times \Delta \times \mathbf{S}$. We let p, q etc. range over $\mathbf{S}$ and a over Δ. For $(p, a, p') \in \longrightarrow$ we write $p \xrightarrow{a} p'$. We also let R, S, X range over binary relations on $\mathbf{S}$.

2.1. Progression and Respectfulness.

Definition 1 (Progress). We say that R progresses to S, denoted $R \rightarrowtail S$, if for all $(p, q) \in R$:

   (1) $\forall p', a.\ p \xrightarrow{a} p' \implies \exists q'.\ q \xrightarrow{a} q' \wedge (p', q') \in S$, and
   (2) $\forall q', a.\ q \xrightarrow{a} q' \implies \exists p'.\ p \xrightarrow{a} p' \wedge (p', q') \in S$

The connection with bisimilarity is that a bisimulation is a relation that progresses to itself, and $\dot{\sim}$ is the union of all bisimulations, but we shall formally not need these notions in our proofs below.

Lemma 1. For all relations R', R, S, S',
$$R' \subseteq R \wedge S \subseteq S' \wedge R \rightarrowtail S \implies R' \rightarrowtail S'$$

Proof. Immediate from Definition 1. $\qquad\qquad\square$

Definition 2 (Respectfulness). A function $\mathcal{F}$ on relations is called *respectful* if, for all relations R and S,
$$R \subseteq S \wedge R \rightarrowtail S \implies \mathcal{F}(R) \subseteq \mathcal{F}(S) \wedge \mathcal{F}(R) \rightarrowtail \mathcal{F}(S)$$

It is known [San98] that the pointwise union of arbitrarily many respectful functions is respectful. Thus, there is a largest respectful function, namely the pointwise union of all respectful functions. Our main result below is to provide an alternative characterization of this function.

2.2. Stratification of Bisimilarity.
We will use Greek letters α, β, ... to range over the ordinals, and reserve λ to stand for a limit ordinal. The following definition is due to Milner [Mil89], originally with the intention of connecting bisimilarity to logical formulas.

Definition 3 (Stratifications of Bisimilarity). For every ordinal α, define a relation $\sim_\alpha$ as follows:
$$\sim_0 := \mathbf{S} \times \mathbf{S}$$
$$\sim_{\alpha+1} := \bigcup \{X \mid X \rightarrowtail \sim_\alpha\}$$
$$\sim_\lambda := \bigcap_{\alpha < \lambda} \sim_\alpha$$

The first clause can be regarded as a special case of the clause for limit ordinals (an intersection of zero sets is the universal relation), but it is clearer to write out the base case explicitly. Milner's idea is that as α increases, $\sim_\alpha$ corresponds to finer behavioural equivalences. Thus $p \sim_0 q$ always holds, $p \sim_1 q$ means that p and q can mimic each other for one transition, $p \sim_2 q$ that they can mimic each other for two transitions, and so on. If $\longrightarrow$ is finitely branching, i.e., for each state the set of outgoing transitions is finite, this sequence converges at ω, i.e., $\sim_\omega = \sim_{\omega+1}$. In general this is not the case. The construction

in the following lemma is illuminating, and we have not seen it explicitly stated before, although we shall not need it for our main result.

Lemma 2. For any ordinal α there exists a transition system where $\sim_\alpha$ and $\sim_{\alpha+1}$ are distinct.

Proof. We will only need transition systems with a single transition label, which we elide. For any ordinal α let the transition system T_α contain as states all ordinals less than or equal to α, and let the transitions be defined by ordinal membership, i.e., $\alpha \longrightarrow \beta$ if $\alpha > \beta$. Since the transitions from a state will be the same in all T_α where the state occurs, we may elide explicit references to the transition systems below. We can now prove the following for all ordinals α, β, γ with $\alpha < \beta$:

$$\alpha \sim_\gamma \beta \text{ iff } \gamma \leq \alpha$$

The proof is by transfinite induction over γ. The cases where $\gamma = 0$ or a limit are immediate. Assume $\gamma = \gamma' + 1$. For the direction $\Leftarrow$, the only transitions from α or β that cannot be mimicked directly (i.e., leading to exactly the same state) are $\beta \longrightarrow \beta'$ for some $\beta' \geq \alpha$. A simulating transition is then $\alpha \longrightarrow \gamma'$ and by induction $\gamma' \sim_{\gamma'} \beta'$. Conversely, assume $\alpha \sim_{\gamma'+1} \beta$ and consider the transition $\beta \longrightarrow \alpha$. Since α can simulate there is a transition $\alpha \longrightarrow \alpha'$ such that $\alpha' \sim_{\gamma'} \alpha$. By induction $\gamma' \leq \alpha'$ whence $\gamma \leq \alpha$.

To conclude the proof of the lemma, take $T_{\alpha+1}$ where it holds that $\alpha \sim_\alpha \alpha + 1$ and $\alpha \not\sim_{\alpha+1} \alpha + 1$. $\qquad\square$

We now establish that the relations $\sim_\alpha$ indeed become smaller as α increases:

Lemma 3. For all ordinals α, $\sim_{\alpha+1} \subseteq \sim_\alpha$.

Proof. By transfinite induction over α.
  (1) $\alpha = 0$: Trivially $\sim_1 \subseteq \mathbf{S} \times \mathbf{S} = \sim_0$.
  (2) $\alpha = \beta + 1$: The induction hypothesis entails $\sim_{\beta+1} \subseteq \sim_\beta$. Hence for any relation X, $X \rightarrowtail \sim_{\beta+1}$ implies $X \rightarrowtail \sim_\beta$ by Lemma 1. Thus $\sim_{\alpha+1} = \bigcup\{X \mid X \rightarrowtail \sim_{\beta+1}\} \subseteq \bigcup\{X \mid X \rightarrowtail \sim_\beta\} = \sim_{\beta+1} = \sim_\alpha$.
  (3) $\alpha = \lambda$: For all $\beta < \lambda$, $\sim_\lambda \subseteq \sim_\beta$ is immediate from Definition 3. As in the successor case, this implies $\sim_{\lambda+1} \subseteq \sim_{\beta+1}$. Moreover, $\sim_{\beta+1} \subseteq \sim_\beta$ by the induction hypothesis. Thus $\sim_{\lambda+1} \subseteq \sim_\beta$ for all $\beta < \lambda$, and therefore $\sim_{\lambda+1} \subseteq \bigcap_{\beta<\lambda} \sim_\beta = \sim_\lambda$.

$\qquad\square$

Lemma 4. For all ordinals α and β, $\alpha \leq \beta \implies \sim_\beta \subseteq \sim_\alpha$.

Proof. By transfinite induction over β, using Lemma 3 for the successor case. The limit case is immediate from Definition 3. $\qquad\square$

2.3. **Convergence of the Stratification.** As demonstrated in Lemma 2, there is no universal ordinal to which the series of equivalences $\sim_\alpha$ converges in all transition systems. For the rest of this paper we assume some fixed transition system. The following lemmas then establish that no matter what this transition system is, there exists an ordinal where the series converges.

Lemma 5. There exists an ordinal ε such that $\sim_{\varepsilon+1} = \sim_\varepsilon$.

The lemma is an instance of a well-known fixed point result for monotone functions. Our proof follows Rubin and Rubin [RR63].

Proof. By Hartogs's theorem [Har15] we can find an ordinal κ such that there is no injection from κ into the powerset of $\mathbf{S} \times \mathbf{S}$. Therefore, there exist ordinals $\varepsilon < \beta < \kappa$ such that $\sim_\varepsilon = \sim_\beta$. Lemma 4 then implies $\sim_{\varepsilon+1} = \sim_\varepsilon$. $\square$

In the following we write ε for the least ordinal provided by Lemma 5, i.e., such that $\sim_{\varepsilon+1} = \sim_\varepsilon$.

Lemma 6. For all ordinals α, $\alpha \geq \varepsilon \implies \sim_\alpha = \sim_\varepsilon$.

Proof. By transfinite induction over α, using Lemma 5 for the successor case. The limit case follows from the induction hypothesis and Lemma 4. $\square$

2.4. **Progression of Strata.** We next show that $\sim_\varepsilon$ progresses to itself.

Lemma 7. For all ordinals α, $\sim_{\alpha+1} \rightarrowtail \sim_\alpha$.

Proof. Suppose $p \sim_{\alpha+1} q$. By Definition 3, $(p, q) \in X$ for some X such that $X \rightarrowtail \sim_\alpha$. $\square$

Lemma 8. $\sim_\varepsilon \rightarrowtail \sim_\varepsilon$

Proof. From Lemmas 7 and 5. $\square$

Lemma 8 proves that $\sim_\varepsilon$ is included in bisimilarity, which is defined as the union of all relations progressing to themselves. The opposite inclusion, which we will not need in this note, is a straightforward exercise (again using transfinite induction over α).

2.5. **The Largest Respectful Function.** We can now give an explicit characterization of the largest respectful function:

Definition 4 (LRF).
$$\text{LRF}(R) := \bigcap \{\sim_\alpha \mid R \subseteq \sim_\alpha\}$$

In other words, by Lemma 4, $\text{LRF}(R)$ is $\sim_\alpha$ for the smallest $\sim_\alpha$ containing all of R. In particular, if $R \subseteq \sim_\varepsilon$ then $\text{LRF}(R) = \sim_\varepsilon$. The crucial properties of LRF are:

Lemma 9. LRF is monotone, i.e., for all relations R and S,
$$R \subseteq S \implies \text{LRF}(R) \subseteq \text{LRF}(S)$$

Proof. Suppose $R \subseteq S$. Then $S \subseteq \sim_\alpha$ implies $R \subseteq \sim_\alpha$. Hence
$$\text{LRF}(R) = \bigcap \{\sim_\alpha \mid R \subseteq \sim_\alpha\} \subseteq \bigcap \{\sim_\alpha \mid S \subseteq \sim_\alpha\} = \text{LRF}(S)$$
$\square$

Theorem 1. LRF is respectful.

Proof. Suppose $R \subseteq S$ and $R \rightarrowtail S$. Then $\mathrm{LRF}(R) \subseteq \mathrm{LRF}(S)$ by Lemma 9. It remains to show $\mathrm{LRF}(R) \rightarrowtail \mathrm{LRF}(S)$. We consider two cases.

Case 1: $S \subseteq \sim_\alpha$ for all ordinals α. Then in particular $S \subseteq \sim_\varepsilon$. Thus also $R \subseteq \sim_\varepsilon$, and $\mathrm{LRF}(R) = \sim_\varepsilon \rightarrowtail \sim_\varepsilon = \mathrm{LRF}(S)$ by Lemma 8.

Case 2: There exists an ordinal α such that $S \not\subseteq \sim_\alpha$. Since the ordinals are well-ordered, we may wlog. assume that α is minimal, i.e., $S \subseteq \sim_\beta$ for all $\beta < \alpha$. Note that $\alpha \neq 0$ since $S \subseteq \mathbf{S} \times \mathbf{S} = \sim_0$. Moreover, α is not a limit ordinal, since $S \subseteq \sim_\beta$ for all $\beta < \lambda$ implies $S \subseteq \bigcap_{\beta<\lambda} \sim_\beta = \sim_\lambda$. Thus $\alpha = \gamma + 1$ for some ordinal γ.

From $R \rightarrowtail S$ and $S \subseteq \sim_\gamma$ we have $R \rightarrowtail \sim_\gamma$ by Lemma 1. Hence $R \subseteq \bigcup\{X \mid X \rightarrowtail \sim_\gamma\} = \sim_{\gamma+1}$. Therefore $\mathrm{LRF}(R) = \bigcap\{\sim_\beta \mid R \subseteq \sim_\beta\} \subseteq \sim_{\gamma+1}$.

Lemma 4 implies $S \not\subseteq \sim_\beta$ for all $\beta \geq \alpha$. Thus $\mathrm{LRF}(S) = \bigcap\{\sim_\beta \mid S \subseteq \sim_\beta\} = \bigcap_{\beta<\alpha} \sim_\beta = \bigcap_{\beta\leq\gamma} \sim_\beta = \sim_\gamma$, again using Lemma 4.

$\mathrm{LRF}(R) \rightarrowtail \mathrm{LRF}(S)$ now follows from $\sim_{\gamma+1} \rightarrowtail \sim_\gamma$ (Lemma 7) and Lemma 1. $\qquad\square$

Finally we establish that LRF is the largest respectful function in the sense that it contains all respectful functions:

Theorem 2. If $\mathcal{F}$ is respectful, then $\mathcal{F}(R) \subseteq \mathrm{LRF}(R)$ for every relation R.

Proof. Suppose $\mathcal{F}$ is respectful. We prove

$$\forall R.\ R \subseteq \sim_\alpha \Longrightarrow \mathcal{F}(R) \subseteq \sim_\alpha$$

for all ordinals α by transfinite induction. ($\mathcal{F}(R) \subseteq \mathrm{LRF}(R)$ then follows from the definition of LRF.)

(1) $\alpha = 0$: Trivially $\mathcal{F}(R) \subseteq \mathbf{S} \times \mathbf{S} = \sim_0$.
(2) $\alpha = \beta + 1$: Suppose $R \subseteq \sim_{\beta+1}$. By Lemma 7, $\sim_{\beta+1} \rightarrowtail \sim_\beta$. Thus $R \rightarrowtail \sim_\beta$ by Lemma 1. Moreover, $\sim_{\beta+1} \subseteq \sim_\beta$ by Lemma 3. Hence $R \subseteq \sim_\beta$, and $\mathcal{F}(R) \rightarrowtail \mathcal{F}(\sim_\beta)$ follows from Definition 2. The induction hypothesis (applied to $\sim_\beta$) implies $\mathcal{F}(\sim_\beta) \subseteq \sim_\beta$. Hence $\mathcal{F}(R) \rightarrowtail \sim_\beta$ by Lemma 1. Therefore $\mathcal{F}(R) \subseteq \bigcup\{X \mid X \rightarrowtail \sim_\beta\} = \sim_{\beta+1}$.
(3) $\alpha = \lambda$: Suppose $R \subseteq \sim_\lambda = \bigcap_{\beta<\lambda} \sim_\beta$. Then $R \subseteq \sim_\beta$ for all $\beta < \lambda$, and the induction hypothesis implies $\mathcal{F}(R) \subseteq \sim_\beta$. Hence $\mathcal{F}(R) \subseteq \bigcap_{\beta<\lambda} \sim_\beta = \sim_\lambda$.

$\qquad\square$

3. Generalisation

We have established that Definition 4 defines the largest respectful function. Our constructions and proofs are quite general and transfer smoothly to other settings. We can for example recast our result in a setting of complete lattices as follows. Assume that a set A with order $\leq$ is a complete lattice with top element $\top$. Let a, b range over A. For a binary relation R on A, the pre-image of b, i.e., $\{a \mid a\,R\,b\}$, is written $R\,b$.

Definition 5 (cf. [Pou07, Definition 1.16])**.** A *progression* is a binary relation R on A such that:

(1) $\leq \circ\, R \circ \leq\ \subseteq\ R$
(2) $\forall b \in A.\ \bigvee R\,b \in R\,b$

As an example, Lemma 1 establishes that $\rightarrowtail$ satisfies condition (1). Moreover, it is obvious from Definition 1 that for any relation S, $\{R \mid R \rightarrowtail S\}$ is closed under arbitrary union. Hence $\rightarrowtail$ also satisfies (2).

Definition 6. For a binary relation R on A, say that a function $f \colon A \to A$ is R-*monotone* if it is monotone with respect to $\leq \cap R$, i.e., $a \leq b \wedge a \, R \, b \Longrightarrow f(a) \leq f(b) \wedge f(a) \, R \, f(b)$.

For example, by Definition 2 respectfulness is $\rightarrowtail$-monotonicity for the lattice of binary relations under the inclusion order.

Definition 7. Given a relation R, for every ordinal α define $z_\alpha \in A$ by the following transfinite induction:
$$\begin{aligned} z_0 &:= \top \\ z_{\alpha+1} &:= \bigvee R \, z_\alpha \\ z_\lambda &:= \bigwedge \{z_\alpha \mid \alpha < \lambda\} \end{aligned}$$

Theorem 3. If R is a progression, the unique largest R-monotone function is
$$\lambda x. \ \bigwedge \{z_\alpha \mid x \leq z_\alpha\}$$

The proof follows the previous section closely, with an arbitrary progression instead of $\rightarrowtail$. The one point of deviation is in the proof of the counterpart of Lemma 7 above, that $z_{\alpha+1} \, R \, z_\alpha$. Here we use condition (2) of Definition 5 to show that $\bigvee R \, z_\alpha$, i.e., $z_{\alpha+1}$, must lie in $R \, z_\alpha$.

As a special case we can consider A to be the lattice of binary relations on states ordered by inclusion, and R to be the progression relation $\rightarrowtail$; we then recover the theorems of the previous section.

Respectful vs. compatible. Pous has developed a theory of up-to techniques based on *compatible* rather than respectful functions. Definition 5 is equivalent to Definition 1.16 in [Pou07]. To a progression R Pous associates the monotone function $s_R \colon A \to A$ given by $\lambda x. \ \bigvee R \, x$. Conversely, given a monotone function $s \colon A \to A$, the set $\{(a, b) \mid a \leq s(b)\}$ defines a progression. A monotone function $f \colon A \to A$ is s-*compatible* if $f \circ s \leq s \circ f$ pointwise.

According to Proposition 1.17(ii) in the same paper, a monotone function $f \colon A \to A$ is s_R-compatible (for a progression R) iff, for all $a, b \in A$, $a \, R \, b$ implies $f(a) \, R \, f(b)$. Note that $R \cap \leq$ is a progression whenever R is a progression. It follows that for monotone functions, R-monotonicity (Definition 6) is exactly $s_{R \cap \leq}$-compatibility.

We did not restrict ourselves to monotone functions in this note, but since the largest respectful function is in fact monotone (cf. Lemma 9), Theorem 3 thus gives the largest $s_{R \cap \leq}$-compatible function for any progression R.

A thorough analysis and comparison between respectfulness and compatibility is in [Pou16], Section 9. In general, R-monotonicity and s_R-compatibility are not equivalent. For monotone functions, R-monotonicity is strictly weaker than s_R-compatibility. However, these differences turn out to be irrelevant when we consider the largest function, allowing us to establish a more direct connection between respectfulness and compatibility.

Theorem 4. If R is a progression, the unique largest s_R-compatible function is
$$\lambda x. \ \bigwedge \{z_\alpha \mid x \leq z_\alpha\}$$

The proof is a minor adaptation of the proof for Theorem 3. In particular, to prove that $\lambda x. \ \bigwedge \{z_\alpha \mid x \leq z_\alpha\}$ is s_R-compatible, we note that $a \, R \, b$ and $b \leq z_\alpha$ implies $a \, R \, z_\alpha$

by progression, hence $a \leq \bigvee R\, z_\alpha = z_{\alpha+1} \leq z_\alpha$. It is then straightforward to adjust the (generalized) proofs of Theorems 1 and 2 to use s_R-compatibility instead of R-monotonicity.

As a corollary of this explicit characterization, we recover [Pou16, Proposition 9.1]: the largest respectful function and the largest compatible function coincide.

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