Topic 8: Inference & Search in CP & LCG
(Version of 22nd September 2021)

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Sweden

Course 1DL441:
Combinatorial Optimisation and Constraint Programming,
whose part 1 is Course 1DL451:
Modelling for Combinatorial Optimisation
Outline

1. Annotations

2. Inference Annotations for CP & LCG

3. Search Annotations for CP & LCG

4. Case Studies
   Balanced Incomplete Block Design
   Warehouse Location
   Sport Scheduling
Outline

1. Annotations

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4. Case Studies
   - Balanced Incomplete Block Design
   - Warehouse Location
   - Sport Scheduling
Annotations:

- Annotations provide information to the backend or to the MiniZinc-to-FlatZinc compiler.
- Annotations are optional.
- A backend may ignore any of the annotations.
- The compiler may introduce further annotations.
- Annotations are attached with `::` to model items.
- Annotations do not affect the model semantics.

Annotations to a constraint:

- Annotations can suggest a propagator to use for the constraint by a CP or LCG backend: see slide 8.

Annotations to the objective:

- Annotations can suggest a search strategy to use by a CP or LCG backend: see slide 14.
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Domains (reminder)

**Definition**

The domain of a variable \( v \), denoted here by \( \text{dom}(v) \), is the set of values that \( v \) can still take during search:

- The domains of the variables are reduced by search and by inference (see the next two slides).
- A variable is said to be fixed if its domain is a singleton.
- Unsatisfiability occurs if a variable domain goes empty.

Note the difference between:

- a domain as a technology-independent declarative entity at the modelling level; and
- a domain as a procedural data structure for CP solving.
CP Solving (reminder)

Tree Search, upon initialising each domain as in the model:

**Satisfaction problem:**

1. Perform inference (see the next slide).
2. If the domain of some variable is empty, then backtrack.
3. If all variables are fixed, then we have a solution.
4. Select a non-fixed variable $v$, partition its domain into two parts $\pi_1$ and $\pi_2$, and make two branches: one with $v \in \pi_1$, and the other one with $v \in \pi_2$.
5. Recursively explore each of the two branches.

**Optimisation problem:** when a feasible solution is found at step 3, first add the constraint that the next solution must be better and then backtrack.
CP Inference

Definition

A propagator for a predicate $\gamma$ deletes from the current domains of the variables of a $\gamma$-constraint the values that cannot be part of a solution to that constraint.

Not all impossible values need to be deleted:

- A domain-consistency propagator deletes all impossible values from the domains.
- A bounds-consistency propagator only deletes all impossible min and max values from the domains.

There exist other, unnamed consistencies for propagators. There is a trade-off between the time & space complexity of a propagator and its achieved deletion of domain values.
Example (Linear equality constraints)

Consider the linear constraint $3 \times x + 4 \times y = z$
with $\text{dom}(x) = 0..1 = \text{dom}(y)$ and $\text{dom}(z) = 0..10$:

- A bounds-consistency propagator reduces $\text{dom}(z)$ to $0..7$.
- A domain-consistency propagator reduces $\text{dom}(z)$ to $\{0, 3, 4, 7\}$.

Time complexity:

- A bounds-consistency propagator for a linear equality constraint can be implemented to run in $O(n)$ time, where $n$ is the number of variables in the constraint.
- A domain-consistency propagator for a linear equality constraint can be implemented to run in $O(n \cdot d^2)$ time, where $n$ is the number of variables in the constraint and $d$ is the sum of their domain sizes, hence in time pseudo-polynomial = exponential in input magnitude.
Controlling the CP Inference

The choice of the right propagator for each constraint may be critical for performance.

Each CP solver and LCG solver has a default propagator for each available constraint predicate.

It is possible to override the defaults with annotations:

- :: domain asks for a domain-consistency propagator.
- :: bounds asks for a bounds-consistency propagator.

Annotations may be ignored, only partially followed, or just approximated: annotations are just suggestions.
Example (n-Queens)

1 array[1..n] of var 1..n: Row;
2 constraint alldifferent(Row) :: domain;
3 constraint alldifferent
4 ([ Row[c]+c | c in 1..n]) :: domain;
5 constraint alldifferent
6 ([ Row[c]−c | c in 1..n]) :: domain;

Test results with Gecode (CP) to first solution for n=101:

<table>
<thead>
<tr>
<th>Inference</th>
<th># nodes</th>
<th>seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>default (no annotation)</td>
<td>348,193</td>
<td>5.5</td>
</tr>
<tr>
<td>bounds on alldifferent</td>
<td>348,193</td>
<td>5.5</td>
</tr>
<tr>
<td>domain on alldifferent</td>
<td>209,320</td>
<td>3.2</td>
</tr>
</tbody>
</table>
Example (n-Queens)

```plaintext
1 array[1..n] of var 1..n: Row;
2 constraint alldifferent(Row) :: domain;
3 constraint alldifferent
4   ([(Row[c]+c)::bounds | c in 1..n]) :: domain;
5 constraint alldifferent
6   ([(Row[c]-c)::bounds | c in 1..n]) :: domain;
```

Test results with Gecode (CP) to first solution for n=101:

<table>
<thead>
<tr>
<th>inference</th>
<th># nodes</th>
<th>seconds</th>
</tr>
</thead>
<tbody>
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<td>default (no annotation)</td>
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<td>5.5</td>
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<tr>
<td>bounds on alldifferent</td>
<td>348,193</td>
<td>5.5</td>
</tr>
<tr>
<td>domain on alldifferent</td>
<td>209,320</td>
<td>3.2</td>
</tr>
<tr>
<td>+ bounds on the linear constraints</td>
<td>&gt; 20M</td>
<td>&gt; 600.0</td>
</tr>
<tr>
<td>bounds on all the constraints</td>
<td>&gt; 20M</td>
<td>&gt; 600.0</td>
</tr>
</tbody>
</table>

Asking for bounds consistency on the implicit linear equality constraints backfires here, as each is on only 2 variables, but it may pay off upon more variables (and be default then).
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Search Strategies

Search Strategies:

- On which variable to branch next?
- How to partition the domain of the chosen variable?
- Which search (depth-first, breadth-first, . . . ) to use?

The search is usually depth-first left-to-right search.

One can suggest to a CP or LCG backend on which variable to branch and how, by making an annotation with:

- a variable selection strategy, and
- a domain partitioning strategy.

A search annotation is sometimes exploited for MIP solvers.
Variable Selection Strategy

The variable selection strategy has an impact on the size of the search tree, especially if the constraints are processed with propagation at every node of the search tree, or if the whole search tree is explored: for example, when it is an optimisation problem or when there are no solutions.

Example (Impact of the variable selection strategy)

Consider \texttt{var 1..2: x, var 1..4: y, var 1..6: z}, branching on all domain values, but no constraints:

- If selecting the variables in the order \texttt{x, y, z}, then the CP search tree has \(1 + 2 + 2 \cdot 4 + 2 \cdot 4 \cdot 6 = 59\) nodes and \(2 \cdot 4 \cdot 6 = 48\) leaves.

- If selecting the variables in the order \texttt{z, y, x}, then the CP search tree has \(1 + 6 + 6 \cdot 4 + 6 \cdot 4 \cdot 2 = 79\) nodes and also \(6 \cdot 4 \cdot 2 = 48\) leaves.
Definition (First-Fail Principle)

To succeed, first try where you are most likely to fail. In practice:

- Select a variable with the smallest current domain.
- Select a variable involved in the largest number of constraints.
- Select a variable recently causing the most backtracks.

Example (Impact of the variable selection strategy)

Finding the first solution to 101-queens with Gecode (CP):

<table>
<thead>
<tr>
<th>search</th>
<th># nodes</th>
<th>seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>default (no annotation)</td>
<td>348,193</td>
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</tr>
<tr>
<td>first_fail</td>
<td>323,275</td>
<td>5.3</td>
</tr>
<tr>
<td>anti_first_fail</td>
<td>&gt; 20M</td>
<td>&gt; 600.0</td>
</tr>
<tr>
<td>input_order</td>
<td>&gt; 13M</td>
<td>&gt; 600.0</td>
</tr>
</tbody>
</table>

(Continued on slide 18)
Domain Partitioning Strategy

The domain partitioning strategy has an impact on the size of the search tree when optimising, when only searching for the first solution, or when performing incomplete search (say when using a time-out).

Example (Impact of the domain partitioning strategy)

Consider \( \text{var } 1..2: x, \text{var } 1..4: y, \text{var } 1..6: z \), domain consistency for \( x \cdot y = z, x \neq y, x \neq z, \text{and } y \neq z \), smallest-domain variable selection, and depth-first search:

- If the domain is split into singletons by increasing order, then 6 CP nodes are explored before finding a solution.
- If the domain is split into singletons by decreasing order, then only 2 CP nodes (the root and a leaf) are explored before finding the solution, without backtracking.
Definition (Best-First Principle)

First try a domain part that is most likely, if not guaranteed, to have values that lead to solutions. This may be like how one would make the greedy choice in a greedy algorithm for the problem at hand, considering its objective function.

Example (Impact of the domain partitioning strategy)

(Continued from slide 16)
Finding the first solution to 101-queens with Gecode (CP):

<table>
<thead>
<tr>
<th>search</th>
<th># nodes</th>
<th>seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>default (no annotation)</td>
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<td>5.5</td>
</tr>
<tr>
<td>first_fail, indomain_min</td>
<td>348,193</td>
<td>5.6</td>
</tr>
<tr>
<td>first_fail, indomain</td>
<td>323,275</td>
<td>5.3</td>
</tr>
<tr>
<td>first_fail, indomain_median</td>
<td>96</td>
<td>0.1</td>
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</table>
Motivation for First-Fail and Best-First

<table>
<thead>
<tr>
<th>Finding a solution</th>
<th>Detecting unsatisfiability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable selection</td>
<td>Must consider all the remaining variables</td>
</tr>
<tr>
<td>Domain partitioning</td>
<td>Need <strong>not</strong> consider all the remaining values: try and find a solution a.s.a.p.</td>
</tr>
</tbody>
</table>

1 Based on material by Yves Deville and Pascal Van Hentenryck
A brancher \textit{int\_search}(X, \phi, \psi) selects a non-fixed variable in the array \(X\) of integer decision variables, using as variable selection strategy \(\phi\) one of the following:

- **input\_order**: select the next variable by order in \(X\)
- **first\_fail**: select a variable with smallest domain
- **smallest**: select a variable with smallest minimum
- **largest**: select a variable with largest maximum
- **occurrence**: select a variable involved in the largest number of active propagators
- **most\_constrained**: use \textit{first\_fail} and break ties with \textit{occurrence}
- **max\_regret**: select a variable with the largest difference between its two smallest domain values

\[\ldots\] (see the \textit{MiniZinc Handbook})

Ties are broken by the order in \(X\). (Continued on next slide)
Then, for the chosen variable, say $v$, the \textit{brancher} selects values in $\text{dom}(v) = \{d_1, \ldots, d_n\}$, with $n \geq 2 \land d_1 < \cdots < d_n$, and builds \textit{guesses}, which are constraints, using as domain partitioning strategy $\psi$ one of the following:

- \texttt{indomain}: branch left-to-right on $v = d_1, \ldots, v = d_n$
- \texttt{indomain\_min}: branch left on $v = d_1$, right on $v \neq d_1$
- \texttt{indomain\_middle}: select $d_i$ nearest $\hat{m} = \lfloor (d_1 + d_n)/2 \rfloor$ and branch left on $v = d_i$, right on $v \neq d_i$
- \texttt{indomain\_median}: select median $d_i = d_{\lfloor (n+1)/2 \rfloor}$ and branch left on $v = d_i$, right on $v \neq d_i$
- \texttt{indomain\_split}: branch left on $v \leq \hat{m}$, right $v > \hat{m}$
- \texttt{indomain\_reverse\_split}: left $v > \hat{m}$, right $v \leq \hat{m}$
- \texttt{outdomain\_random}: select a random value $d_i$ and branch left on $v \neq d_i$, right on $v = d_i$
- \ldots (see the \texttt{MiniZinc Handbook})
Definition (Boolean Brancher)

A brancher `bool_search(X, \phi, \psi)` selects a non-fixed variable in the array `X` of Boolean decision variables, using variable selection strategy `\phi` and domain partitioning strategy `\psi`, with the same choices as for integer variables, under the convention `false < true`.

Definition (Chaining of Branchers)

A brancher `seq_search([\beta_1, \ldots, \beta_n])` chains branchers `\beta_1, \ldots, \beta_n`: when brancher `\beta_i` is finished, branch with `\beta_{i+1}`.

Careful: A search annotation goes between the `solve` and `satisfy`, `minimize`, or `maximize` keywords, and it is ignored elsewhere. See the example on slide 37.

The search strategy of Gecode for the variables not in the search annotation depends also on the `output` statement.
Definition

A set (decision) variable takes a set as value, and has a set of sets as domain. For its domain to be finite, a set variable must be a subset of a given finite set $\Sigma$.

Integers are totally ordered, but sets are partially ordered: propagation for set variables is harder. Also, set domains can get huge: $\mathcal{O}(2^{\left|\Sigma\right|})$. A trade-off is to over-approximate the domain of a set variable $S$ by a pair $\langle \ell, u \rangle$ of finite sets, denoting the set of all sets $\sigma$ such that $\ell \subseteq \sigma \subseteq u \subseteq \Sigma$:

- $\ell$ is the current set of mandatory elements of $S$;
- $u \setminus \ell$ is the current set of optional elements of $S$.

Example

The domain of a set var represented as $\langle \{1\}, \{1, 2, 3, 4\} \rangle$ has the sets $\{1\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{1, 2, 3, 4\}$. Deleting $\{1, 2, 3\}$ is impossible!
Definition (Set Brancher)

A \textit{brancher} set\textunderscore search\((X, \phi, \psi)\) selects a non-fixed variable \(S \doteq \langle \ell, u \rangle\) in the array \(X\) of set variables, using a variable selection strategy \(\phi\) on slide 20:

- \textbf{first\_fail}: select a variable with smallest \(|u \setminus \ell|\)
- \textbf{smallest}: select a variable with smallest \(\min(u \setminus \ell)\)
- \ldots (see the \textit{MiniZinc} Handbook)

Then, for the chosen variable, say \(S \doteq \langle \ell, u \rangle\), it selects an element in \(u \setminus \ell = \{d_1, \ldots, d_n\}\), with \(d_1 < \cdots < d_n\), and adds guesses using a domain partitioning strategy \(\psi\) on slide 21:

- \textbf{indomain\_min}: branch left on \(d_1 \in S\), right on \(d_1 \notin S\)
- \textbf{outdomain\_max}: left on \(d_n \notin S\), right on \(d_n \in S\)
- \textbf{outdomain\_median}: select median \(d_i = d_{\lfloor (n+1)/2 \rfloor}\) and branch left on \(d_i \notin S\), right on \(d_i \in S\)
- \ldots (see the \textit{MiniZinc} Handbook)
Designing Search Strategies

Problem-specific strategies:
Beside general principles (first-fail and best-first), there are often good strategies that can be designed using problem-specific knowledge. In MiniZinc, it is often easy to express such strategies in terms of problem-specific concepts.

Interaction with symmetry-breaking constraints:
For higher solving speed, suggest a domain partitioning that drives the search towards solutions satisfied by the symmetry-breaking constraints.

Counter-example
For $a + b + c = 38$, with all variables in 1..19, and $\text{symmetry\_breaking\_constraint}(a<b \ \land \ b<c)$, do not use $\text{int\_search}([a,b,c],\text{input\_order},\text{indomain\_max})$. 
Interaction with the choice of dummy values:

For higher solving speed, suggest a domain partitioning that drives the search towards trying the dummy values (recall the examples of Topic 4: Modelling) at the right moment.

Example (Student Seating, viewpoint 2 revisited again)

```plaintext
int: dummyS = 0; % Advice: also experiment with nStudents+1
set of int: StudentsAndDummy = 1..nStudents union {dummyS};
% Student[c] = the student, possibly dummy, on chair c:
array[1..nChairs] of var StudentsAndDummy: Student;
constraint global_cardinality_closed(Student,
    [dummyS] ++ [i | i in 1..nStudents],
    [nChairs - nStudents] ++ [1 | i in 1..nStudents]);
...
```

Under Gecode default search, \( \text{dummy}_S = 0 \) is a lot slower than \( \text{dummy}_S = \text{nStudents}+1 \), whose performance can however be matched for \( \text{dummy}_S = 0 \) by suggesting, say, \text{int_search} (Student, \text{first_fail}, \text{indomain_max})\): one should only try and seat a dummy student on a chair after it turns out that no real student can be seated on it.
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   Warehouse Location
   Sport Scheduling
Agricultural experiment design, AED

<table>
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<th>plot2</th>
<th>plot3</th>
<th>plot4</th>
<th>plot5</th>
<th>plot6</th>
<th>plot7</th>
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<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Constraints to be satisfied:

1. Equal growth load: Every plot grows 3 grains.
2. Equal sample size: Every grain is grown in 3 plots.
3. Balance: Every grain pair is grown in 1 common plot.

Instance: 7 plots, 7 grains, 3 grains/plot, 3 plots/grain, balance 1.

General term: balanced incomplete block design (BIBD).
The following constraints (of Topic 5: Symmetry) break the full row and column symmetries, but not their compositions:

4. \[
\text{constraint symmetry\_breaking\_constraint(}
\begin{align*}
&\text{forall}(v \text{ in Varieties} \text{ diff } \{\text{max(Varieties)}\})(
\begin{align*}
&\text{lex\_greater(BIBD}[v,..],\text{BIBD}[v+1,..])))
\end{align*}
\end{align*}
\]

5. \[
\text{constraint symmetry\_breaking\_constraint(}
\begin{align*}
&\text{forall}(b \text{ in Blocks} \text{ diff } \{\text{max(Blocks)}\})(
\begin{align*}
&\text{lex\_greater\_req(BIBD}[..,b],\text{BIBD}[..,b+1])))
\end{align*}
\end{align*}
\]

The use of \text{lex\_greater\_req} (as opposed to \text{lex\_lesseq}, say) is justified by the following search strategy:

- All \text{BIBD}[v, b] variables have the same 0..1 domain, so the first-fail principle cannot distinguish between them: let us fill the \text{BIBD} incidence matrix in input order (left-to-right in each row, and top-down across rows).

- Since typically fewer 1s than 0s occur in a \text{BIBD}, the best-first principle suggests trying 1 before 0:

\[
\text{:: int\_search(BIBD,input\_order,indomain\_max)}
\]
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The Warehouse Location Problem (WLP)

A company considers opening warehouses at some candidate locations in order to supply its existing shops:

- Each candidate warehouse has the same maintenance cost.
- Each candidate warehouse has a supply capacity, which is the maximum number of shops it can supply.
- The supply cost to a shop depends on the warehouse.

Determine which candidate warehouses actually to open, and which of them supplies which shops, so that:

1. Each shop is supplied by exactly one actually opened warehouse.
2. Each actually opened warehouse supplies a number of shops at most equal to its capacity.
3. The sum of the actually incurred maintenance costs and supply costs is minimal.
WLP: Sample Instance Data

\[ \text{Shops} = \{\text{Shop}_1, \text{Shop}_2, \ldots, \text{Shop}_{10}\} \]

\[ \text{Whs} = \{\text{Berlin, London, Ankara, Paris, Rome}\} \]

\[ \text{maintCost} = 30 \]

<table>
<thead>
<tr>
<th></th>
<th>Berlin</th>
<th>London</th>
<th>Ankara</th>
<th>Paris</th>
<th>Rome</th>
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<tr>
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<td>4</td>
<td>2</td>
<td>1</td>
<td>3</td>
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</table>

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<th>Paris</th>
<th>Rome</th>
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<td>74</td>
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<td>Shop_3</td>
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<td>97</td>
<td>71</td>
<td>96</td>
<td>70</td>
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<td>2</td>
<td>55</td>
<td>73</td>
<td>69</td>
<td>61</td>
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<table>
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<th>Paris</th>
<th>Rome</th>
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<tbody>
<tr>
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<td>47</td>
<td>65</td>
<td>55</td>
<td>71</td>
<td>95</td>
</tr>
</tbody>
</table>
WLP Model 1: Variables (Reminder)

Automatic enforcement of the total-function constraint (1):

\[ \text{Supplier} = \begin{bmatrix} \text{Shop}_1 & \text{Shop}_2 & \cdots & \text{Shop}_{10} \\ \in \text{Whs} & \in \text{Whs} & \cdots & \in \text{Whs} \end{bmatrix} \]

\( \text{Supplier}[s] \) denotes the supplier warehouse for shop \( s \).

Variables redundant with \( \text{Supplier} \), but not mutually:

\[ \text{Open} = \begin{bmatrix} \text{Berlin} & \text{London} & \text{Ankara} & \text{Paris} & \text{Rome} \\ \in 0..1 & \in 0..1 & \in 0..1 & \in 0..1 & \in 0..1 \end{bmatrix} \]

\( \text{Open}[w] = 1 \) if and only if (iff) warehouse \( w \) is opened.
WLP Model 1: Annotations

The capacity constraint, now boosted with the inference annotation :: domain, and the channelling constraint of Supplier with Open are as in Topic 6: Case Studies.

Let the new decision variable Cost[s] represent the actual supply cost for shop s. It is non-mutually redundant with Supplier[s] and has the one-way channelling constraint:
forall(s in Shops) (Cost[s] = SupplyCost[s, Supplier[s]])

The objective becomes:
minimize maintCost * sum(Open) + sum(Cost)

For shop s, let dom(Cost[s]) = \{d_1, d_2, \ldots, d_n\}, with \( n \geq 2 \land d_1 < d_2 < \cdots < d_n\): the regret of shop s is \( d_2 - d_1 \), that is the difference in supply cost between its currently cheapest and second-cheapest potential suppliers.
The maximal-regret strategy recommends:

- **Variable selection:**
  Select a decision variable \( \text{Cost}[s] \) such that the shop \( s \) currently has the maximal regret.

- **Value selection and guesses:**
  Select the smallest value \( d \) in \( \text{dom}(\text{Cost}[s]) \).
  Branch left on \( \text{Cost}[s] = d \), right on \( \text{Cost}[s] \neq d \).

The \( \text{Supplier}[s] \) decision variables are then branched on by increasing order of \( s \) and by increasing value. This brancher accelerates search only if, for some shops \( s \), some values in \( \text{SupplyCost}[s, \ldots] \) are equal.

Upon one-way channelling from \( \text{Supplier} \) to \( \text{Open} \), the \( \text{Open}[w] \) decision variables are then branched on by increasing order of \( w \) and by increasing value, in order to fix any still non-fixed variables to 0 faster than by relying upon minimisation (like in Topic 6: Case Studies).
This search strategy is expressed in MiniZinc as follows:

```
1 solve
2  :: seq_search([ 
3      int_search(Cost,max_regret,indomain_min),
4      int_search(Supplier,input_order,indomain_min),
5      int_search(Open,input_order,indomain_min)
6  ])
7 minimize maintCost * sum(Open) + sum(Cost)
```

Objective values, upon the three seen ways of channelling, within 35 seconds by Gecode (CP) on a MacBook-Air laptop, on a hard instance with 16 warehouses of capacity 4 supplying 50 shops, of minimal cost at most 1,190,733:

<table>
<thead>
<tr>
<th>search</th>
<th>Model 1</th>
<th>Model 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>one-way</td>
<td>two-way</td>
</tr>
<tr>
<td>default (no annotation)</td>
<td>none</td>
<td>1,869,494</td>
</tr>
<tr>
<td>first_fail on Supplier</td>
<td>1,520,326</td>
<td>1,524,034</td>
</tr>
<tr>
<td>first_fail on Cost</td>
<td>1,218,079</td>
<td>1,223,704</td>
</tr>
<tr>
<td>max_regret on Cost</td>
<td>1,193,637</td>
<td>1,198,276</td>
</tr>
</tbody>
</table>
Outline

1. Annotations

2. Inference Annotations for CP & LCG

3. Search Annotations for CP & LCG

4. Case Studies
   Balanced Incomplete Block Design
   Warehouse Location
   Sport Scheduling
The Sport Scheduling Problem (SSP)

Find schedule in $\text{Periods} \times \text{Weeks} \rightarrow \text{Teams} \times \text{Teams}$ for

- $|\text{Teams}| = n$ and $n$ is even
- $|\text{Weeks}| = n-1$
- $|\text{Periods}| = n/2$ periods per week

subject to the following constraints:

1. Each possible game is played exactly once.
2. Each team plays exactly once per week.
3. Each team plays at most twice per period.

Idea for a model, and a solution for $n=8$

<table>
<thead>
<tr>
<th></th>
<th>Wk 1</th>
<th>Wk 2</th>
<th>Wk 3</th>
<th>Wk 4</th>
<th>Wk 5</th>
<th>Wk 6</th>
<th>Wk 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>P 1</td>
<td>1 vs 2</td>
<td>1 vs 3</td>
<td>2 vs 6</td>
<td>3 vs 5</td>
<td>4 vs 7</td>
<td>4 vs 8</td>
<td>5 vs 8</td>
</tr>
<tr>
<td>P 2</td>
<td>3 vs 4</td>
<td>2 vs 8</td>
<td>1 vs 7</td>
<td>6 vs 7</td>
<td>6 vs 8</td>
<td>2 vs 5</td>
<td>1 vs 4</td>
</tr>
<tr>
<td>P 3</td>
<td>5 vs 6</td>
<td>4 vs 6</td>
<td>3 vs 8</td>
<td>1 vs 8</td>
<td>1 vs 5</td>
<td>3 vs 7</td>
<td>2 vs 7</td>
</tr>
<tr>
<td>P 4</td>
<td>7 vs 8</td>
<td>5 vs 7</td>
<td>4 vs 5</td>
<td>2 vs 4</td>
<td>2 vs 3</td>
<td>1 vs 6</td>
<td>3 vs 6</td>
</tr>
</tbody>
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- $|\text{Teams}| = n$ and $n$ is even
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- $|\text{Periods}| = n/2$ periods per week

subject to the following constraints:

1. Each possible game is played exactly once.
2. Each team plays exactly once per week.
3. Each team plays at most twice per period.

Idea for a model, and a solution for $n=8$, with a dummy week $n$ of duplicate games:

<table>
<thead>
<tr>
<th></th>
<th>Wk 1</th>
<th>Wk 2</th>
<th>Wk 3</th>
<th>Wk 4</th>
<th>Wk 5</th>
<th>Wk 6</th>
<th>Wk 7</th>
<th>Wk 8</th>
</tr>
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<tr>
<td>P 1</td>
<td>1 vs 2</td>
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<tr>
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<td>3 vs 4</td>
<td>2 vs 8</td>
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</tr>
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<td>2 vs 7</td>
<td>2 vs 4</td>
</tr>
<tr>
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<td>7 vs 8</td>
<td>5 vs 7</td>
<td>4 vs 5</td>
<td>2 vs 4</td>
<td>2 vs 3</td>
<td>1 vs 6</td>
<td>3 vs 6</td>
<td>1 vs 8</td>
</tr>
</tbody>
</table>
SSP Model 1: Variables (reminder)

A 3d matrix $\text{Team}[\text{Periods}, \text{ExtendedWeeks}, \text{Slots}]$ of variables in $\text{Teams}$, denoted $T$ below, over a schedule extended by a dummy week where teams play fictitious duplicate games in the period where they would otherwise play only once, thereby transforming constraint (3) into:

(3’) Each team plays exactly twice per period.

$\text{Team} =$

<table>
<thead>
<tr>
<th></th>
<th>Wk 1</th>
<th></th>
<th></th>
<th>Wk $n-1$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P 1</td>
<td>one</td>
<td></td>
<td>two</td>
<td></td>
<td>one</td>
</tr>
<tr>
<td>:</td>
<td>$\in T$</td>
<td>$\in T$</td>
<td>...</td>
<td>...</td>
<td>$\in T$</td>
</tr>
<tr>
<td>P $n/2$</td>
<td>one</td>
<td></td>
<td>two</td>
<td></td>
<td>one</td>
</tr>
<tr>
<td>:</td>
<td>$\in T$</td>
<td>$\in T$</td>
<td>...</td>
<td>...</td>
<td>$\in T$</td>
</tr>
</tbody>
</table>

$\text{Team}[p, w, s]$ is the numeric name of the team that plays in period $p$ of week $w$ in game slot $s$. 
SSP Model 1: More Variables (reminder)

Declare a 2d matrix $\text{Game} [\text{Periods, Weeks}]$ of decision variables in $\text{Games}$ over the non-extended weeks:

\[
\text{Game} =
\begin{array}{ccc}
\text{Period 1} & \text{Week 1} & \cdots & \text{Week } n-1 \\
\vdots & \in \text{Games} & \cdots & \in \text{Games} \\
\text{Period } n/2 & \in \text{Games} & \cdots & \in \text{Games} \\
\end{array}
\]

$\text{Game} [p, w]$ is the game played in period $p$ of week $w$.

The 2d $\text{Game}$ is mutually redundant with the first $n - 1$ 2d columns of the 3d $\text{Team}$, which is over the extended weeks.
SSP Model 1: Channelling Constraint

Two-way channelling constraint (reminder):

\[
\text{forall}(p \text{ in Periods}, w \text{ in Weeks}) \\
(Team[p,w,\text{one}] \times n + Team[p,w,\text{two}] = Game[p,w])
\]

The game number in \text{Game} of each period and week corresponds to the teams scheduled at that time in \text{Team}.

If a CP or LCG solver cannot enforce domain consistency on linear equality, even when :: \text{domain} is used, then precompute a \text{table} constraint:

\[
\text{forall}(p \text{ in Periods}, w \text{ in Weeks}) \\
(\text{table} \{[Team[p,w,\text{one}],Team[p,w,\text{two}],Game[p,w]], \\
\text{array2d}(1..(n*(n-1) \div 2), 1..3, \\
[[f,s,f*n+s][i] | f,s \text{ in Teams where } f<s, i \text{ in } 1..3]})
\]

% \{1,2,6|1,3,7|1,4,8|2,3,11|2,4,12|3,4,16\} for n=4
SSP Model 1: Search Annotation

It suffices to follow the first-fail principle:

- **Variable selection:**
  Select a decision variable $\text{Game}[p,w]$ with the currently smallest domain.

- **Value selection and guesses:**
  Select the smallest value $d$ in $\text{dom}(\text{Game}[p,w])$.
  Branch left on $\text{Game}[p,w] = d$ and right on $\text{Game}[p,w] \neq d$.

The $\text{Team}[p,w,s]$ variables need no brancher as they take their values fast through either the 2-way channelling constraint, especially if propagated to domain consistency, or the $\text{global_cardinality_closed}(\ldots)$ formulation of constraint (3’) in Topic 6: Case Studies. This search strategy is expressed in MiniZinc as follows:

```mini
:: int_search(Game, first_fail, indomain_min)
```
SSP Model 2: Smaller Domains for Game

A round-robin schedule suffices to break many of the remaining symmetries:

- Restrict the games of the first week to the set \( \{1 \text{ vs } 2\} \cup \{t + 1 \text{ vs } n + 2 - t \mid 1 < t \leq n/2\} \)
- For the remaining weeks, transform each game \( f \text{ vs } s \) of the previous week into a game \( f' \text{ vs } s' \), where

\[
f' = \begin{cases} 
1 & \text{if } f = 1 \\
2 & \text{if } f = n \\
f + 1 & \text{otherwise}
\end{cases}
\quad \text{and} \quad
s' = \begin{cases} 
2 & \text{if } s = n \\
s + 1 & \text{otherwise}
\end{cases}
\]

We must determine the period of each game, not its week!

**Search strategy:**
Choose games for the first period across all the weeks, then for the first week across all the remaining periods, then for the next period across all the remaining weeks, then for the next week across all the remaining periods, etc.
Interested in More Details?

For more details on WLP & SSP and their strategies, see:

Van Hentenryck, Pascal. 
*The OPL Optimization Programming Language.*

Van Hentenryck, Pascal. 
*Constraint and integer programming in OPL.*

Van Hentenryck, Pascal; Michel, Laurent; Perron, Laurent; and Régis, Jean-Charles. 
*Constraint programming in OPL.*