Topic 4: Modelling (for CP & LCG)\(^1\)
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Modelling for Combinatorial Optimisation

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Outline

1. Viewpoints & Dummy Values
2. Implied Constraints
3. Redundant Variables & Channelling Constraints
4. Pre-Computation
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1. Viewpoints & Dummy Values

2. Implied Constraints

3. Redundant Variables & Channelling Constraints

4. Pre-Computation
Recap

1 Modelling: express problem in terms of
   - parameters,
   - decision variables,
   - constraints, and
   - objective.

2 Solving: solve using a state-of-the-art solver.
Example (Student Seating Problem)

Given:

- \( n_{\text{Students}} \) students,
- \( n_{\text{Pgms}} \) study programmes, and
- \( n_{\text{Chairs}} \) chairs around tables,

find a seating arrangement such that:

1. each table has students of distinct study programmes;
2. each table has either at least half its chairs occupied, or none;
3. a maximum number of student preferences on being seated at the same table are satisfied.

\[
\begin{align*}
\text{nStudents} &= 15, \\
\text{nPgms} &= 3, \\
\text{nChairs} &= 20 \geq \text{nStudents}, \\
\text{Tables} &= [1..4, \ldots, 17..20]
\end{align*}
\]

What are suitable decision variables for this problem?
A viewpoint is a choice of decision variables.

Example (Student Seating Problem)

**Viewpoint 1:** Which chair does each student sit on?

1. % Chair[s] = the chair of student s:
2. array[1..nStudents] of var 1..nChairs: Chair;
3. constraint all_different(Chair); % max 1 student per chair

**Viewpoint 2:** Which student, if any, sits on each chair?

1. int: dummyS = 0; % Advice: also experiment with nStudents+1
2. set of int: StudentsAndDummy = 1..nStudents union {dummyS};
3. % Student[c] = the student, possibly dummy, on chair c:
4. array[1..nChairs] of var StudentsAndDummy: Student;
5. constraint global_cardinality_closed(Student,
   [dummyS] ++ [i | i in 1..nStudents],
   [nChairs - nStudents] ++ [1 | i in 1..nStudents]);
   % all_different(Student) if nStudents+1..nChairs are dummy

We revisit this problem at slide 19 and the choice of dummy values in Topic 5: Symmetry as well as in Topic 8: Inference & Search in CP & LCG.

Let us see how viewpoints differ when stating constraints.
Example (Objects, Shapes, and Colours)

There are $n$ objects, $s$ shapes, and $c$ colours, with $s \geq n$. Assign a shape and a colour to each object such that:

1. the objects have distinct shapes;
2. the numbers of objects of the used colours are distinct;
3. other constraints, yielding NP-hardness and distinguishing objects and shapes, are satisfied.

This problem can be modelled from different viewpoints:

1. Which colour, if any, does each shape have?
2. Which shapes, if any, does each colour have?
3. Which shape and colour does each object have?
4. . .

Each viewpoint comes with benefits and drawbacks.
Example (Objects, Shapes, and Colours)

Viewpoint 1: Which colour, if any, does each shape have?

1 int: n; % number of objects
2 int: s; % number of shapes
3 constraint assert(s >= n, "Not enough shapes");
4 int: c; % number of colours
5 int: dummyColour = 0; % Advice: also experiment with c+1
6 set of int: ColoursAndDummy = 1..c union {dummyColour};
7 % Colour[i] = colour, possibly dummy, of object of shape i:
8 array[1..s] of var ColoursAndDummy: Colour;
9 % There are n objects:
10 constraint count(Colour,dummyColour) = s - n;
11 % The numbers of objects of the used colours are distinct:
12 constraint all_different_except(global_cardinality(Colour,1..c),{0});
13 % The objects have distinct shapes:
14 % implied by lines 6 and 8!
15 % ... add here the other constraints ...
16 solve satisfy;

So what are the shape and colour of a particular object?!
☞ Map the objects onto the shapes with non-dummy colour!
Example (Objects, Shapes, and Colours)

Viewpoint 2: Which shapes, if any, does each colour have?

1 int: n; % number of objects
2 int: s; % number of shapes
3 constraint assert(s >= n, "Not enough shapes");
4 int: c; % number of colours
5 % Shapes[i] = the set of shapes of colour i:
6 array[1..c] of var set of 1..s: Shapes;
7 % There are n objects:
8 % implied by line 12 below!
9 % The numbers of objects of the used colours are distinct:
10 constraint all_different_except(
    [card(Shapes[colour]) | colour in 1..c], {0});
11 % The objects have distinct shapes:
12 constraint n = card(array_union(Shapes));
13 % ... add here the other constraints ...
14 solve satisfy;

Post-process: map the objects onto actually used shapes.
Can we also model this viewpoint without set variables?
☞ Yes, see the next slide!
Example (Objects, Shapes, and Colours)

Viewpoint 2: Which shapes, if any, does each colour have?

1 int: n; % number of objects
2 int: s; % number of shapes
3 constraint assert(s >= n, "Not enough shapes");
4 int: c; % number of colours
5 % NbrObj[i,j] = the number of objects of colour i & shape j:
6 array[1..c,1..s] of var 0..1: NbrObj;
7 % There are n objects:
8 constraint n = sum(NbrObj);
9 % The numbers of objects of the used colours are distinct:
10 constraint all_different_except(
   [sum(NbrObj[colour,..]) | colour in 1..c], {0});
11 % The objects have distinct shapes:
12 constraint forall(shape in 1..s)(sum(NbrObj[..,shape])<=1);
13 % ... add here the other constraints ...
14 solve satisfy;

Which model for viewpoint 2 is clearer or better?
☞ Ask and try!
Example (Objects, Shapes, and Colours)

Viewpoint 3: Which shape & colour does each object have?

1 int: n; % number of objects
2 int: s; % number of shapes
3 constraint assert(s >= n, "Not enough shapes");
4 int: c; % number of colours
5 array[1..n] of var 1..s: Shape; % Shape[i] = shape of obj. i
6 array[1..n] of var 1..c: Colour; % Colour[i] = colour of i
7 % There are n objects:
8 % implied by lines 5 and 6!
9 % The numbers of objects of the used colours are distinct:
10 constraint all_different_except
11   (global_cardinality_closed(Colour,1..c),{0});
12 % The objects have distinct shapes:
13 constraint all_different(Shape);
14 % ... add here the other constraints ...
15 solve satisfy;

We have used two parallel arrays with the same index set but different domains in order to represent pair variables.
Which viewpoint is better in terms of:

- **Size of the search space:**
  - Viewpoint 1: $O((c + 1)^s)$, which is independent of $n$
  - Viewpoint 2: $O(2^{s \cdot c})$, which is independent of $n$
  - Viewpoint 3: $O(s^n \cdot c^n)$

Does this actually matter?

- **Ease of formulating the constraints and the objective:**
  - It depends on the unstated other constraints.
  - Ideally, we want a viewpoint that allows global-constraint predicates to be used.

- **Performance:**
  - Hard to tell: we have to run experiments!

- **Readability:**
  - Who is going to read the model?
  - What is their background?

There are no correct answers here: we actually need to think about this and run experiments.
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Example (The Magic Series Problem)

The element at index \( i \) in \( I = 0..(n-1) \) is the number of occurrences of \( i \). Solution: \( \text{Magic} = [1, 2, 1, 0] \) for \( n=4 \).

Variables: \( \text{Magic} = \begin{array}{cccc}
0 & 1 & \cdots & n-1 \\
\in I & \in I & \cdots & \in I
\end{array} \)

Constraint:

\[
\forall (i \in I) (\text{Magic}[i] = \sum (j \in I) (\text{Magic}[j] = i))
\]

or, logically equivalently but better:

\[
\forall (i \in I) (\text{count} (\text{Magic}, i, \text{Magic}[i]))
\]

or, logically equivalently and even better:

\[
\text{global_cardinality_closed}(
\text{Magic}, \text{array1d}(I, [i | i \in I]), \text{Magic})
\]

Implied Constraint:

\[
\sum (\text{Magic}) = n \setminus \sum (i \in I) (\text{Magic}[i] \times i) = n
\]

For \( n=80 \), using a CP solver: only 7 search nodes are explored instead of 302; the solving is 1,000 times faster.
Definition

An implied constraint, also called a redundant constraint, is a constraint that logically follows from other constraints.

Benefit:
Solving may be faster, without losing any solutions. However, not all implied constraints accelerate the solving.

Good practice in MiniZinc:
Flag implied constraints using the `implied_constraint` predicate. This allows backends to handle them differently, if wanted (see Topic 9: Modelling for CBLS):

```
predicate implied_constraint(var bool: c) = c;  VS
predicate implied_constraint(var bool: c) = true;
```

Example

```
constraint implied_constraint(sum(Magic) = n);
```

In Topic 5: Symmetry, we see the equally recommended `symmetry_breaking_constraint` predicate.
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Redundant Decision Variables

Example \((n\text{-queens})\)

Use both the \(n^2\) decision variables \(\text{Queen}[r,c]\) in \(0..1\) and the \(n\) decision variables \(\text{Row}[c]\) in \(1..n\).

Definition

A redundant decision variable represents information that is already available via some other decision variables. We distinguish mutual and non-mutual redundancy.

Benefit: Easier modelling, or faster solving, or both.

Examples (see Topic 6: Case Studies)

- Each \(\text{Queen}[,c]\) is mutually redundant with \(\text{Row}[c]\).
- Best model of Black-Hole Patience: mutual redundancy
- Models 1 & 3 of Warehouse Location: non-mutual red.
Channelling Constraints

Example (n-queens)

One-way channelling from the $n$ decision variables $\text{Row}[c]$ in $1..n$ to the $n^2$ decision variables $\text{Queen}[r,c]$ in $0..1$:

constraint $\forall (c \in 1..n) (\text{Queen}[\text{Row}[c],c] = 1)$

Definition

A channelling constraint helps establish the coherence of the value of a variable that is redundant with other variables.

Examples (see Topic 6: Case Studies)

- Best model of Black-Hole Patience: 2-way channelling.
- Models 1 & 3 of Warehouse Location: 1-way vs 2-way.
- Experiment with channelling between the viewpoints for the Objects, Shapes, and Colours problem (slide 7).
Example (Student Seating, viewpoint 2 revisited)

```plaintext
1 int: dummyS = 0; % Advice: also experiment with nStudents+1
2 set of int: StudentsAndDummy = 1..nStudents union {dummyS};
3 % Student[c] = the student, possibly dummy, on chair c:
4 array[1..nChairs] of var StudentsAndDummy: Student;
5 constraint global_cardinality_closed(Student,
   [dummyS] ++ [i | i in 1..nStudents],
   [nChairs - nStudents] ++ [1 | i in 1..nStudents]);
6 int: dummyP = 0; % Advice: also experiment with nPgms+1
7 set of int: PgmsAndDummy = 1..nPgms union {dummyP};
8 % Pgm[s] = the given study programme of student s:
9 array[1..nStudents] of 1..nPgms: Pgm;
10 % Programme[c] = the programme of the student on chair c
   (non-mutually redundant with Student):
11 array[1..nChairs] of var PgmsAndDummy: Programme;
12 % 1-way channelling from Student to Programme, for dummyS=0:
13 constraint forall(c in 1..nChairs) (Programme[c] =
   array1d(StudentsAndDummy, [dummyP]++Pgm)[Student[c]]);
14 % (1) Each table has students of distinct study programmes:
15 constraint forall(T in Tables)
   (all_different_except([Programme[c] | c in T]), {dummyP});
16 ... % constraint (2) and objective (3) of slide 5
```

Note that Student uniquely determines Programme, but not vice-versa: one can also formulate (1) with Student.
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Example (Prize-Pool Division)

Consider a maximisation problem where the objective function is the division of an unknown prize pool by an unknown number of winners:

```c
... 
array[1..5] of int: Pools = [1000,5000,15000,20000,25000];
var 1..5: x; % index of the actual prize pool within Pools
var 1..500: nbrWinners; % the number of winners
constraint ... x ... nbrWinners ...;
solve maximize Pools[x] div nbrWinners; % implicit: element!
```

**Observation:** We should beware of using the `div` function on decision variables, because:

- It yields weak inference, at least in CP & LCG solvers.
- Its inference takes unnecessary time and memory.

**Idea:** We can pre-compute all possible objective values.
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**Example (Prize-Pool Division, revisited)**

Pre-compute a 2d array, indexed by $1..5$ and $1..500$, for each possible value pair of $x$ and $\text{nbrWinners}$:

1. 
2. `array[1..5] of int: Pools = [1000,5000,15000,20000,25000];`
3. `var 1..5: x; % index of the actual prize pool within Pools`
4. `var 1..500: nbrWinners; % the number of winners`
5. `constraint ... x ... nbrWinners ...;`
6. `array[1..5,1..500] of int: ObjVal = array2d(1..5,1..500, [Pools[p] div n | p in 1..5, n in 1..500]);`
7. `solve maximize ObjVal[x,nbrWinners]; % implicit: 2d-element!`