Topic 3: Constraint Predicates
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Course 1DL442:
Combinatorial Optimisation and Constraint Programming,
whose part 1 is Course 1DL451:
Modelling for Combinatorial Optimisation

¹Many thanks to Guido Tack for feedback
Outline

1. Motivation
2. all_different
3. nvalue
4. global_cardinality
5. element
6. bin_packing
7. knapsack
8. cumulative, disjunctive
9. circuit, subcircuit
10. lex_lesseq
11. regular, table
12. Modelling Checklist
Outline

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Motivation

**Examples**

Let $A$ be an array of variables:

- An `all_different(A)` constraint holds if and only if (iff) all the elements of $A$ take different values:

  $\forall i,j \in \text{index}_\text{set}(A) \text{ where } i<j \quad (A[i] \neq A[j])$

- A `count(A,v) >= c` constraint holds if and only if the count of occurrences in $A$ of $v$ is at least $c$, where $v$ and $c$ can be variables:

  $\sum(i \in \text{index}_\text{set}(A))(A[i]=v) \geq c$
**Definition**

A definition of a constraint predicate is its semantics, stated in MiniZinc in terms of usually simpler constraint predicates.

**Examples**

See some MiniZinc-provided default definitions at slide 4.

**Definition**

Each use of a predicate is decomposed during flattening by inlining either its MiniZinc-provided default definition or an overriding backend-provided solver-specific definition.

**Examples**

If a predicate $\gamma$ on arguments $X$ is supported by a solver, then its backend provides $\gamma(X) = \gamma(X)$ as specific definition.
Motivation:

+ More compact and intuitive models, because more expressive predicates are available: islands of common combinatorial structure are identified in declarative high-level abstractions.

+ Faster solving, due to better inference and relaxation, enabled by more global information in the model, provided the predicate is a built-in of the used solver.

Enabling constraint-based modelling:

- Constraint predicates over any number of variables go by many names: global-constraint predicates, combinatorial-constraint predicates, ...


- Some predicates cannot be reified, say via bool2int.
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The **all_different** Predicate

**Definition (Laurière, 1978)**

An `all_different(A)` constraint holds iff all the elements of the array `A` of variables take different values.

Its default definition in MiniZinc is a conjunction of \( \frac{n(n-1)}{2} \) disequality constraints when `A` has `n` elements:

\[
forall(i, j \in \text{index\_set}(A) \text{ where } i<j)(A[i] \neq A[j])
\]

An `all_different_except(A, S)` constraint allows multiple occurrences of the exception values in the set `S`.

**Examples**

- **n-Queens problem**: see Topic 1: Introduction.
- **Photo problem**: see Topic 2: Basic Modelling.
- **Student Seating problem**: see Topic 4: Modelling.
- **Object, Shapes, and Colours**: see Topic 4: Modelling.
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The \text{nvalue} Predicate

\textbf{Definition (Pachet and Roy, 1999)}

An \text{nvalue}(m,A) constraint holds if and only if variable $m$ takes the number of distinct values taken by the elements of the array $A$ of variables, say $1d$ and with indices $1..n$:

$$|\{A[1], \ldots, A[n]\}| = m$$

The expression \text{nvalue}(A) denotes the number of distinct values taken by the elements of the array $A$ of variables.

If $|A| = n$ then \text{nvalue}(n,A) means \text{all\_different}(A). Always use the most specific available constraint predicate!

\textbf{Example}

Model 2 of the Warehouse Location problem:
see Topic 6: Case Studies.
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The `global_cardinality` Predicate

Definition (R´egin, 1996)

A `global_cardinality(A, V, C)` constraint holds iff each variable $C[j]$ has the number of elements of the array $A$ of variables that take value $V[j]$. Variants exist.

Its default definition in MiniZinc includes:

```plaintext
forall(j in index_set(V))(count(A, V[j]) = C[j])
```

It means `all_different(A)` if $V = \bigcup_i \text{dom}(A[i])$ and $\text{dom}(C[j]) = \{0, 1\}$ for each $j$.

Always use the most specific available predicate!

Examples

- Magic Series problem; Student Seating problem;
  Object, Shapes, and Colours: see Topic 4: Modelling.
- Warehouse Location and Sports Scheduling problems: see Topic 6: Case Studies.
A Common Source of Inefficiency in Models

Example

The model snippet

\[
\text{constraint } \forall (j \in \text{index_set}(V)) \ (\text{count}(A,V[j]) = C[j]);
\]

should be reformulated, due to the shared array \(A\), into:

\[
\text{constraint } \text{global_cardinality}(A,V,C);
\]

by applying the default definition backwards:

- at worst, it will be applied forwards while flattening;
- at best, the invoked solver has better inference.

This advice holds for each global-constraint predicate, and for all (quantified) constraints over shared variables.
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The **element** Predicate

**Definition (Van Hentenryck and Carillon, 1988)**

An element \((i, A, e)\) constraint, where:
- \(A\) is an array of variables,
- \(i\) is an integer variable, and
- \(e\) is a variable,

holds if and only if \(A[i] = e\).

For better model readability, the `element` predicate should not be used, as the functional form \(A[\phi]\) is allowed, even if \(\phi\) is an integer expression involving at least one variable.
**Use:** The `element` predicate and its functional form \( A[\phi] \) help model an unknown element of an array.

**Example (Job allocation at minimal salary cost)**

**Given** jobs `Jobs` and the salaries of work applicants `Apps`, **find** a work applicant for each job **such that** some constraints (on the qualifications of the work applicants for the jobs, on workload distribution, etc) are satisfied and the total salary cost is minimal:

```plaintext
array[Apps] of 0..1000: Salary; % Salary[a]/job by a
array[Jobs] of var Apps: Worker; % job j by Worker[j]
solve minimize sum(j in Jobs)(Salary[Worker[j]]);
constraint ...; % qualifications, workload, etc
```

Line 3 is equivalent to the less readable formulation

```plaintext
array[Jobs] of var 0..max(Salary): Cost; % Cost[j] for job j
constraint forall(j in Jobs)
    (element(Worker[j],Salary,Cost[j]));
solve minimize sum(Cost);
```

We do not know at modelling time the worker of each job!
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The `bin_packing_load` Predicate

**Definition**

Let item $i$ have the given weight or volume $V[i]$. Let variable $B[i]$ denote the bin into which item $i$ is put. Let variable $L[b]$ denote the load of bin $b$. A `bin_packing_load(L, B, V)` constraint holds iff each $L[b]$ is the sum of the $V[i]$ where $B[i]$ equals $b$. Variant predicates exist.

**Example (Balanced academic curriculum problem)**

Given, for each course $c$ in `Courses`, a workload $W[c]$ and a set `Pre[c]` of prerequisite courses, find a semester $Sem[c]$ in $1..n$ for each course $c$ in order to satisfy all the prerequisites under a balanced workload:

1. `constraint bin_packing(sum(W) div n, Sem, W);`
2. `constraint forall(c in Courses, p in Pre[c])(Sem[p]<Sem[c]);`
A Common Source of Inefficiency in Models

Example

The model snippet

```plaintext
constraint forall(b in Bins)
  (Load[b] = sum(i in Items where Bin[i]=b)(Vol[i]));
```

should be reformulated, due to the shared array `Bin` and the `where` clause on the variables `Bin[i]`, as follows:

```plaintext
constraint bin_packing_load(Load,Bin,Vol);
```

There are many incarnations of this pattern:

- **Bins = semesters; Items = courses; Vol[i] = credits for course i; Bin[i] = semester of course i; Load[b] = total credits for all courses in semester b;**

- **Bins = staff; Items = tasks; Vol[i] = reward for task i; Bin[i] = employee assigned to task i; Load[b] = income over tasks assigned to employee b.**
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The knapsack Predicate

Definition

Let item type $t$ have the given weight or volume $V[t]$. Let item type $t$ have the given value or profit $P[t]$. Let the variable $X[t]$ denote the number of items of type $t$ that are put into a given knapsack. Let the variables $v$ and $p$ respectively denote the total volume and total profit of what is in the knapsack. Given $n$ item types, a knapsack ($V, P, X, v, p$) constraint holds iff

$$\sum_{t \text{ in } 1..n} (V[t] \times X[t]) = v$$

and

$$\sum_{t \text{ in } 1..n} (P[t] \times X[t]) = p.$$ 

Example

To model the Knapsack Problem for a knapsack of given capacity $c$, add $v \leq c$ and maximize $p$. 
Motivation

all different
nvalue
global cardinality
element
bin_packing
knapsack
cumulative, disjunctive
circuit, subcircuit
lex.lesseq
regular, table
Modelling Checklist

### Example ([https://xkcd.com/287](https://xkcd.com/287))

A simplified version of the Knapsack Problem, but still NP-hard.

```plaintext
1 array[1..6] of int: Cost = [215, 275, 335, 355, 420, 580];
2 array[1..6] of int: Joy = [ 0, 0, 0, 0, 0, 0 ];
3 array[1..6] of var 0..(1505 div min(Cost)): Amount;
4 constraint knapsack(Cost, Joy, Amount, 1505, 0);
5 solve satisfy;
```

See this [interview](https://www.youtube.com/watch?v=example) for some interesting trivia.
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Assume we want to schedule a set of tasks to be performed over a given period such that we have the \textit{earliest} end.

\textbf{Definition}

A task $T_i$ is a triple $\langle S[i], D[i], R[i] \rangle$ of constants or variables, where:

- $S[i]$ is the starting time of task $T_i$
- $D[i]$ is the duration of task $T_i$
- $R[i]$ is the quantity of a global resource needed by $T_i$

Tasks may be run in parallel if the global resource suffices.

Sample schedule with parallel tasks and bounded resource
A precedence constraint of task $T_1$ on task $T_2$ expresses that the performing of $T_1$ must finish before $T_2$ can start. We say that task $T_1$ precedes task $T_2$.

Sample tasks (bubbles), durations (black numbers), resource requirements (blue numbers), and precedences (orange arrows). Task T7 is a dummy task, as we do not know which of tasks T5 and T6 will finish last.
Let us temporarily ignore the bounded global resource: If we have an unlimited global resource or each task has its own local resource, then the polynomial-time-solvable problem of finding the earliest ending time, under only the precedence constraints, for performing all the tasks can be modelled using linear inequalities.

Example (continued)

The precedence constraints indicated by the orange arrows on slide 25 are modelled as follows, based on the task durations indicated there in black:

```plaintext
1. constraint D = [2,1,4,2,3,1,0];
3. % add here the resource constraint of the next slide
4. solve minimize S[7];
```
The **cumulative** Predicate

Definition (Aggoun and Beldiceanu, 1993)

A *cumulative* \((S, D, R, u)\) constraint, where each task \(T_i\) has a starting time \(S[i]\), a duration \(D[i]\), and a resource requirement \(R[i]\), holds if and only if the resource upper limit \(u\) is never exceeded when performing the \(T_i\).

**cumulative** does not ensure any precedence constraints between the tasks: these have to be stated separately.

Example (end)

To ensure that the global resource capacity of \(u = 8\) units, say, is never exceeded under the resource requirements of the tasks indicated in **blue** on slide 25, add the following:

```plaintext
constraint cumulative(S, D, [1, 3, 3, 2, 4, 6, 0], 8);
```
The disjunctive Predicate

Definition

A non-overlap constraint between tasks \( T_1 \) and \( T_2 \) states that either \( T_1 \) precedes \( T_2 \) or \( T_2 \) precedes \( T_1 \), say because both tasks require a resource that is available only for one task at a time. We say that tasks \( T_1 \) and \( T_2 \) do not overlap.

Definition (Carlier, 1982)

A disjunctive \((S, D)\) constraint, where each task \( T_i \) has a starting time \( S[i] \) and a duration \( D[i] \), holds iff no two tasks \( T_i \) and \( T_j \) overlap in time. It is also known as unary.

It has among others the following definitions:

- \[ \text{forall}(i,j \text{ in } 1..n \text{ where } i<j) \]
  \[ ((S[i]+D[i]<=S[j]) \lor (S[j]+D[j]<=S[i])) \]
- \[ \text{cumulative}(S, D, [1 | i \text{ in } 1..n], 1) \]

Always use the most specific available constraint predicate!
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Enabling the representation of a circuit in a digraph:

- Let variable $S[v]$ represent the successor of vertex $v$.
- The domain of $S[v]$ is the set of vertices to which there is an arc from vertex $v$, plus $v$ itself.

Example

Assume the successor variables in $S$ take these values:

- $[b, c, d, a]$: one circuit $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$
- $[c, a, b, d]$: one subcircuit $a \rightarrow c \rightarrow b \rightarrow a$ and $S[d]=d$
- $[a, b, c, d]$: one empty subcircuit: $S[v]=v$ for all $v$ in Vertices
- $[c, d, a, b]$: two subcircuits, namely $a \rightarrow c \rightarrow a$ and $b \rightarrow d \rightarrow b$
- $[b, d, a, d]$: $c \rightarrow a \rightarrow b \rightarrow d$ is not a (sub)circuit
The circuit and subcircuit Predicates

Definition (Laurièrè’78; Beldiceanu & Contejean’94)

A circuit(S) constraint holds iff the arcs \( v \rightarrow S[v] \) form a Hamiltonian circuit: each vertex is visited exactly once.
A subcircuit(S) constraint holds iff circuit(S’) holds for exactly one possibly empty but non-singleton subarray S’ of S, and \( S[v] = v \) for all the other vertices.

Examples (Vehicle routing)

Travelling salesperson problem (generalise this for vehicle routing problems with multiple vehicles or side constraints):

```plaintext
solve minimize sum(c in Cities)(Dist[c, Next[c]]);
constraint circuit(Next);
```

Requiring a directed path from vertex \( v \) to vertex \( w \):

```plaintext
constraint subcircuit(S) \( \setminus \) S[w] = v;
```

upon adding \( v \) to the domain of \( S[w] \) if need be.

Many graph constraints, including dpath, exist in MiniZinc.
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The **lex_lesseq** Predicate

### Example

```
lex_lesseq([1,2,34,5,678], [1,2,36,45,78]),
because 34 < 36, even though 678 ≤ 78.
```

### Definition

A **lex_lesseq**($A, B$) constraint, where $A$ and $B$ are same-length 1d arrays of variables, say both with indices in $1..n$, holds iff $A$ is lexicographically at most equal to $B$:

- either $n=0$, or $A[1]<B[1]$,

Variant predicates exist.

### Usage:

Exploit index symmetries in **matrix models**, where there are arrays of variables:

see Topic 4: Modelling, and see Topic 5: Symmetry.
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Regular Expressions

Examples (Regular Expressions)

- \((0|1)^*0\) denotes the set of even binary numbers.
- \(1^*(011^*)(0|\epsilon)\) denotes the set of strings of zeros and ones without consecutive zeros.
- \((0|1)^*00(0|1)^*\) denotes the set of strings of zeros and ones with consecutive zeros.

Notation for strings:

- Let \(\epsilon\) denote the empty string.
- Let \(v \cdot w\) denote the concatenation of strings \(v\) and \(w\).
- Let \(w^i\) denote the concatenation of \(i\) copies of string \(w\).
Regular Expressions and Languages

Definition

Let \( \Sigma \) be an alphabet, that is a finite set of symbols. Regular expressions over \( \Sigma \) are defined as follows:

- \( \emptyset \) is a regular expression: its language, \( L(\emptyset) \), is \( \emptyset \).
- \( \epsilon \) is a regular expression: \( L(\epsilon) = \{\epsilon\} \).
- If \( \sigma \in \Sigma \), then \( \sigma \) is a regular expression: \( L(\sigma) = \{\sigma\} \).
- If \( r \) and \( s \) are regular expressions, then \( rs \) is a regular expression: \( L(rs) = \{v \cdot w \mid v \in L(r) \land w \in L(s)\} \).
- If \( r \) and \( s \) are regular expressions, then \( r | s \) is a regular expression: \( L(r | s) = L(r) \cup L(s) \).
- If \( r \) is a regular expression, then \( r^* \) is a regular expression: \( L(r^*) = \{w^i \mid i \in \mathbb{N} \land w \in L(r)\} \).

A regular expression defines a regular language over \( \Sigma \).
Regular Expressions

Common abbreviations for regular expressions:
Let $r$ be a regular expression:
- $r?$ denotes $r|\epsilon$; example in MiniZinc syntax: "12?"
- $r^+$ denotes $rr^*$; example in MiniZinc syntax: "34+
- $r^4$ denotes $rrrr$; example in MiniZinc syntax: "56\{4\}"
- $[1\ 2\ 3\ 4]$ denotes $1|2|3|4$; same syntax in MiniZinc
- $[5-8]$ denotes $[5\ 6\ 7\ 8]$; same syntax in MiniZinc
- $[9\-11\ 14]$ denotes $[9\ 10\ 11\ 14]$; same in MiniZinc
- $\ldots$ (see the MiniZinc documentation)

Usage: Regular expressions are good for the specification of regular languages, but not so good for reasoning on them, where one often uses finite automata instead.
Deterministic Finite Automaton (DFA)

Example (DFA for regular expression ss(ts)*|ts(t|ss)*)

Conventions:
- Start state, marked by arc coming in from nowhere: A.
- Accepting states, marked by double circles: D and E.
- Determinism: There is one outgoing arc per symbol in alphabet \( \Sigma = \{s, t\} \); missing arcs go to a non-accepting missing state that has self-loops on every symbol in \( \Sigma \).
The **regular** Predicate

**Definition (Pesant, 2004)**

A $\text{regular (A, Q, S, d, q0, F)}$ constraint holds iff the values taken by the variables of the 1d array $A$ form a string of the regular language accepted by the DFA with states $1..Q$, symbols $1..S$, transition function $d$ in $1..Q \times 1..S \rightarrow 0..Q$ with missing state 0, start state $q0$, and accepting states $F$. A $\text{regular (A, r)}$ constraint holds iff $A$ forms a string of the regular language denoted by the regular expression $r$.

**Example**

The DFA of the previous slide is represented as follows upon encoding the states \{A,B,C,D,E\} as $1..Q$ and the alphabet \{s,t\} as $1..S$: we have $Q=5$ states, $S=2$ symbols, transition function $d=[|2,3|4,0|5,0|0,2|3,5|]$, start state $q0=1$, and accepting states $F={4,5}$. 
The **table** Predicate

### Definition

A *table* \((A, T)\) constraint holds iff the values taken by the 1d variable array \(A\) form a row of the 2d value array \(T\).

The 2d array \(T\) gives an *extensional definition* of a new constraint predicate, as opposed to the *intensional definition* given so far for all other constraint predicates.

### Example

If the variable array, say \(X\), of the *regular(...)* constraint of the previous slide for the DFA of two slides ago has four variables, then that constraint is equivalent to

\[
\text{table}(X, [ | 1,1,2,1 | 2,1,1,1 | 2,1,2,2 | ]).
\]
Example (The Nonogram Puzzle: instance)

Each hint gives the sequence of lengths of blue blocks in its row or column, with at least one white cell between blocks, but possibly none before the first and after the last block.
Example (The Nonogram Puzzle: instance)

Each hint gives the sequence of lengths of blue blocks in its row or column, with at least one white cell between blocks, but possibly none before the first and after the last block.

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Solution:

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**Motivation**
- All different
- Nvalue
- Global cardinality
- Element
- Bin packing
- Knapsack
- Cumulative, disjunctive
- Circuit, subcircuit
- Lex.lesseq
- Regular, Table
Example (The Nonogram Puzzle: model)

Model:

- Variables: An enumeration-type variable for each cell, with value $w$ if it is to be coloured white, and value $b$ if it is to be coloured blue.

- Constraints: State a `regular` constraint for each hint. For example, for a hint 2 3 1 on a row or column $A$ of length $n \geq 8$, state the constraint `regular(A, "w* b{2} w+ b{3} w+ b{1} w*")`.

See Survey of Paint-by-Number Puzzle Solvers: the straightforward model above fares well, at least with a CP solver, compared to hand-written problem-specific code.
Example (Nurse Rostering)

Each nurse is assigned each day to one of the following:

- **r** regular shift (this value is not available on Sundays)
- **e** extended shift (this value is not available on Sundays)
- **s** Sunday shift (this value is only available on Sundays)
- **o** day off

The nurse labour union imposes the following regulations:

- Monday off after a Sunday shift
- No single extended shifts
- One day off after two consecutive extended shifts

For each nurse \( n \), state the following constraint over the scheduling horizon, say 17 weeks here:

\[
\text{regular}(\text{Roster}[n, \text{Sun1..Sat17}], "(s o | e e o | r | o)*)\]

Further, a hospital has constraints on nurse presence.
Example (The Kakuro Puzzle: instance)

Fill in digits of $1 \ldots 9$ such that the digits of each word are pairwise distinct and add up to the number to the left (for horizontal words) or on top (for vertical words) of the word.

\[
\begin{array}{ccc}
11 & 4 & \\
5 & & 10 \\
14 & & \\
17 & & 3 \\
6 & 4 & 3 \\
10 & 3 & \\
3 & & \\
\end{array}
\]
Example (The Kakuro Puzzle: instance)

Fill in digits of 1..9 such that the digits of each word are pairwise distinct and add up to the number to the left (for horizontal words) or on top (for vertical words) of the word.
Example (The Kakuro Puzzle: first model)

Model:

- Variables: A variable for each cell, with domain 1..9.
- Constraints: For each hint $K[\alpha] + \cdots + K[\beta] = \sigma$, state all_different(i in \(\alpha..\beta\))(K[i]) /\ sum(i in \(\alpha..\beta\))(K[i]) = \sigma.

Performance, using a CP solver:

- 22 \times 14 Kakuro with 114 hints: 9638 nodes, 160 s
- 90 \times 124 Kakuro with 4558 hints: ? nodes, ? years

Symptom: The decomposition may give weak inference: for $x \neq y /\ x+y=4$, CP inference gives $x, y \ in 1..3$, not noticing that 2 should be pruned from both domains. We may need a custom predicate all_different_sum, constraining up to 9 variables over the domain 1..9.
Example (The Kakuro Puzzle: second model)

**New model:** Use the regular or table predicate for the all_different and sum-based constraints of each hint?

- For the hint $x+y=4$: regular([x, y], "1 3|3 1").
- For the hint $y+z=3$: regular([y, z], "1 2|2 1").
- One can also use table instead:
  table([x, y], [|1, 3|3, 1|]) /\ table([y, z], [|1, 2|2, 1|]).
- What about the hint $K[\alpha] + \cdots + K[\alpha+8] = 45$?
  There are 9! = 362,880 solutions...
Motivation

Different n-value global cardinality element bin-packing knapsack cumulative, disjunctive circuit, subcircuit lex.lesseq regular, table

Modelling Checklist

Example (The Kakuro Puzzle: second model, end)

New model (end):

- For the hint \( K[\alpha] + \cdots + K[\alpha+8] = 45 \), it suffices to state `all_different(i in \alpha..\alpha+8)(K[i])`, as the sum of 9 distinct non-0 digits is necessarily 45.
- For the hint \( K[\alpha] + \cdots + K[\alpha+7] = \sigma \), it suffices to state `all_different([K[i] | i in \alpha..\alpha+7]+[45-\sigma])`.
- For the hint \( K[\alpha] = \sigma \), it suffices to state \( K[\alpha] = \sigma \).

Other opportunities for improvement exist.

New performance, using a CP solver:

- 22 × 14 Kakuro with 114 hints: 0 search nodes, 28 ms!
- 90 × 124 Kakuro with 4558 hints: 0 nodes, 345 ms!

Published diabolically hard Kakuros (like the 22 × 14 one mentioned above) where the new model pays off are rare.

The Kakuro story is based on material by Christian Schulte.
When to Use These Predicates?

Rapid prototyping of a new constraint predicate:
The *regular* and *table* predicates are very useful in the following conjunctive situation:

- A needed constraint predicate \( \gamma \) on a 1d array of variables is not a built-in of MiniZinc or the used solver.
- A definition of \( \gamma \) in terms of built-in predicates is not obvious to the modeller, or it has turned out that its *inference* is too expensive or too weak.
- The modeller does not have the time or skill to design an inference algorithm for \( \gamma \), or deems \( \gamma \) not reusable.
- The complexity and strength of an inference algorithm for \( \gamma \) are not deemed crucial for the time being.
Important Modelling Device

Example (Encoding a small function)

The constraint $x \times x = y$, where there is exactly one $y$ for every $x$, may yield poor inference: for $x$ in $1..9$, say, try \texttt{element}(x, [d*d | d in 1..9], y), that is $[d*d | d in 1..9][x] = y$, for better inference.

The \texttt{element} predicate is a specialisation of \texttt{regular} and \texttt{table}, just like a function is a special case of a relation.

Example (Encoding a small relation)

The constraint $x \times x = \text{abs}(y)$, where there can be more than one $y$ for every $x$, and vice-versa, may yield poor inference: for $x$ in $0..3$, say, try the less readable \texttt{table}([x, y], [[0, 0], [1, -1], [1, 1], [2, -4], [2, 4], [3, -9], [3, 9]]) for better inference (maybe not with a MIP solver).
Motivation

Checklist

Bibliography

Pesant, Gilles.
A regular language membership constraint for finite sequences of variables.

Hopcroft, John E.; Motwani, Rajeev; Ullman, Jeffrey D.
Outline

1. Motivation
2. all different
3. nvalue
4. global cardinality
5. element
6. bin packing
7. knapsack
8. cumulative, disjunctive
9. circuit, subcircuit
10. lex lesseq
11. regular, table
12. Modelling Checklist
Checklist for Designing or Reading a Model

11. Predicates with the most specific semantics are used
12. Global constraints are not replaced by their definitions
13. Constraints over shared variables are ideally merged
14. The `element` predicate is not used explicitly, for clarity
15. Functions are encoded if needed by implicit `element`
16. Relations are encoded if needed by `regular/table`