The GLT class as a Generalized Fourier Analysis and applications

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Abstract

Recently, the class of Generalized Locally Toeplitz (GLT) sequences has been introduced [17, 18] as a generalization both of classical Toeplitz sequences and of variable coefficient differential operators and, for every sequence of the class, it has been demonstrated that it is possible to give a rigorous description of the asymptotic spectrum [3, 21] in terms of a function (the symbol) that can be easily identified; see also [20].

This generalizes the notion of a symbol for differential operators (discrete and continuous) or for Toeplitz sequences for which it is identified through the Fourier coefficients and is related to the classical Fourier Analysis.

The GLT class has nice algebraic properties and indeed it has been proven that it is stable under linear combinations, products, and inversion when the sequence which is inverted shows a sparsely vanishing symbol (sparsely vanishing symbol = a symbol which vanishes at most in a set of zero Lebesgue measure). Furthermore, the GLT class virtually includes any approximation by local methods (Finite Difference, Finite Element, Isogeometric Analysis [4, 5, 19] etc) of partial differential equations (PDEs) and, based on this, we demonstrate that our results on GLT sequences can be used in a PDE setting in various directions:

1. as a generalized Fourier Analysis for the design and for the study of preconditioned iterative and semi-iterative methods, when dealing with variable coefficients, non rectangular domains, non uniform gridding or triangulations,

2. for a multigrid analysis of convergence and for providing spectral information on large preconditioned systems in the variable coefficient case,

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3. in order to provide a tool for the stability analysis of PDE numerical schemes (e.g. a necessary von Neumann criterium for variable coefficient systems of PDEs is obtained, uniformly with respect to the boundary conditions), etc.

We will discuss specifically problems 1) and 2) and other possible directions in which the GLT analysis can be conveniently employed; see below for a recent bibliography on the subject.

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References

[1] A. Aricò, M. Donatelli, S. Serra-Capizzano. V-cycle optimal convergence for certain (multilevel) structured linear systems. *SIAM J. Matrix Anal. Appl.* **26** (2004), 186–214.

[2] B. Beckermann, S. Serra-Capizzano. On the asymptotic spectrum of Finite Element matrix sequences. *SIAM J. Numer. Anal.* **45** (2007), 746–769.

[3] A. Böttcher, B. Silbermann. *Introduction to Large Truncated Toeplitz Matrices.* Springer-Verlag, New York (1999).

[4] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems.* SIAM (2002).

[5] J.A. Cottrell, T.J.R. Hughes, Y. Bazilevs. *Isogeometric Analysis: Toward Integration of CAD and FEA.* John Wiley & Sons, New York (2009).

[6] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, H. Speleers. Robust and optimal multi-iterative techniques for IgA Galerkin linear systems. *Comput. Methods Appl. Mech. Engrg.* **284** (2015), 230–264.

[7] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, H. Speleers. Robust and optimal multi-iterative techniques for IgA collocation linear systems. *Comput. Methods Appl. Mech. Engrg.* **284** (2015), 1120–1146.

[8] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, H. Speleers. Spectral analysis and spectral symbol of matrices in isogeometric collocation methods. *Math. Comp.* (2015) in press.

[9] A. Dorostkar, M. Neytcheva, S. Serra-Capizzano. Spectral analysis of coupled PDEs and of their Schur complements via the notion of Generalized Locally Toeplitz sequences. Tech. Report 2015-008, Div. Scientific Computing, Uppsala University (2015).

[10] C. Garoni, C. Manni, F. Pelosi, S. Serra-Capizzano, H. Speleers. On the spectrum of stiffness matrices arising from isogeometric analysis. *Numer. Math.* **127** (2014), 751–799.

[11] C. Garoni, C.Manni, S. Serra-Capizzano, D. Sesana, H. Speleers. Spectral analysis and spectral symbol of matrices in isogeometric Galerkin methods. Tech. Report 2015-005, Div. Scientific Computing, Uppsala University (2015).

[12] C. Garoni, S. Serra-Capizzano, D. Sesana. Spectral Analysis and Spectral Symbol of d-variate Q_p Lagrangian FEM Stiffness Matrices. Tech. Report 2014-021, Div. Scientific Computing, Uppsala University (2014); *SIAM J. Matrix Anal. Appl.* (2015) to appear.

[13] C. Garoni, S. Serra-Capizzano, P. Vassalos. A general tool for determining the asymptotic spectral distribution of Hermitian matrix-sequences. *Oper. Matrices* (2015) in press.

[14] L. Golinskii, S. Serra-Capizzano. The asymptotic properties of the spectrum of nonsymmetrically perturbed Jacobi matrix sequences. *J. Approx. Theory* **144** (2007), 84–102.

[15] S. Serra-Capizzano. Matrix algebra preconditioners for multilevel Toeplitz matrices are not superlinear, *Linear Algebra Appl.* **343–344** (2002), 303–319.

[16] S. Serra-Capizzano. Convergence analysis of two-grid methods for elliptic Toeplitz and PDEs matrix-sequences. *Numer. Math.* **92** (2002), 433–465.

[17] S. Serra-Capizzano. Generalized Locally Toeplitz sequences: spectral analysis and applications to discretized partial differential equations. *Linear Algebra Appl.* **366** (2003), 371–402.

[18] S. Serra-Capizzano. The GLT class as a generalized Fourier Analysis and applications. *Linear Algebra Appl.* **419** (2006), 180–233.

[19] J.C. Strikwerda. *Finite Difference Schemes and Partial Differential Equations.* SIAM (2004).

[20] P. Tilli. Locally Toeplitz sequences: spectral properties and applications. *Linear Algebra Appl.* **278** (1998), 91–120.

[21] E.E. Tyrtyshnikov. A unifying approach to some old and new theorems on distribution and clustering. *Linear Algebra Appl.* **232** (1996) 1–43.