UQ3

Representation of random inputs (§5)

parameter $\alpha(x,\omega)$ in

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\alpha(x,\omega)\frac{\partial T}{\partial x}\right) + f(t,x)$$

$\alpha(x,\omega)$ infinite-dimensional

approximate by finite-dimensional expansion

$Q = (Q_1, \cdots Q_p)$ p mutually independent

$Q_i(\omega) : \Omega \to \mathbb{R}$ with density $\rho_{Q_i}(q_i)$

independent $\Rightarrow \rho_Q(q) = \prod_{i=1}^{p} \rho_{Q_i}(q_i)$

Approximate $\alpha(x,\omega)$ by

$$\alpha(x,\omega) \approx \bar{\alpha}(x) + \sum_{n=1}^{N} Q_n(\omega)\phi_n(x)$$

We want:

Q_n mutually independent

small N, densities $\rho_{Q_n}(q_n)$

$\bar{\alpha}(x) = \mathbb{E}[\alpha(x,\omega)]$, $\phi_n(x)$ basis functions

$Q = (Q_1, \cdots Q_n)$ mutually indep. random var.

Karhunen-Loève expansions (PCA, principal component analysis, POD, proper orthogonal decomposition)

$$\alpha(x,\omega) = \bar{\alpha}(x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n}\,\phi_n(x)\,Q_n(\omega)$$

$\mathbb{E}(\alpha(x,\omega)) = \bar{\alpha}(x)$, $\mathrm{Cov}(\alpha(x,\omega), \alpha(y,\omega)) = C(x,y)$

correlation in space

λ_n, ϕ_n eigenvalues and orthonormal eigenfunctions of C

$$\int_D C(x,y)\phi_n(y)\,dy = \lambda_n \phi_n(x)$$

$$Q_n(\omega) = \frac{1}{\sqrt{\lambda_n}} \int_D (\alpha(x,\omega) - \bar{\alpha}(x))\phi_n(x)\,dx$$

$$\mathbb{E}(Q_n) = 0, \quad \mathbb{E}(Q_m Q_n) = \delta_{mn}$$

Truncated K-L expansion for numerical purposes

$$\alpha(x,\omega) = \bar{\alpha}(x) + \sum_{n=1}^{N} \sqrt{\lambda_n}\, \phi_n(x)\, Q_n(\omega)$$

Parameter selection techniques §6

- Identifiable parameters

$$y = f(q), \quad q = (q_1, q_2, \dots q_p)$$

q is identifiable at q^* if $f(q) = f(q^*) \Rightarrow q = q^*$ for any admissible $q \in Q$

q is identifiable with respect to space $I(q)$ if this holds for all $q^* \in I(q)$ (identifiable subspace)

unidentifiable subspace $NI(q)$ is orthogonal complement to $I(q)$ ($x \in NI(q) \Rightarrow x \perp I(q)$)

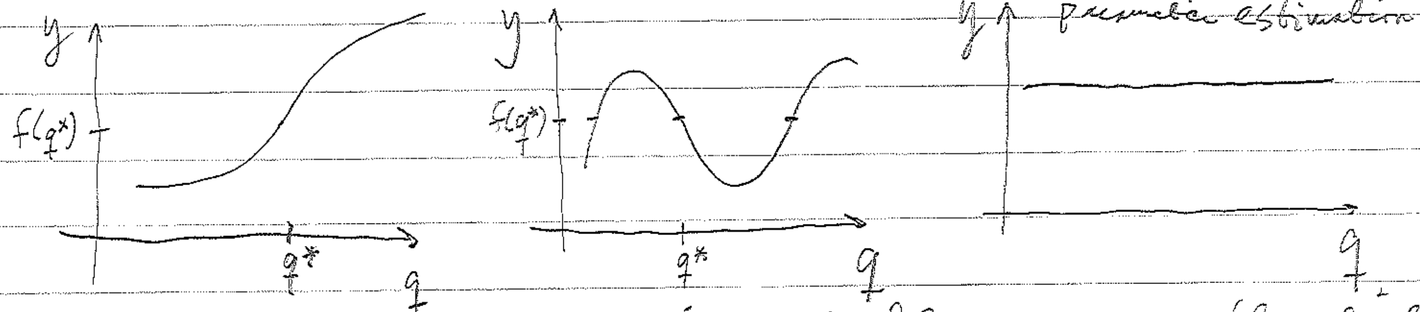
$\Rightarrow Q = I(q) \oplus NI(q)$

- Influential parameters

q is noninfluential if $\forall q$ and $q^* \in NI(q)$

$|y(q) - y(q^*)| < \varepsilon$ for all q and $q^* \in NI(q)$

can be fixed in parameter estimation



q identifiable $\qquad$ q nonidentifiable $\qquad$ q noninfluential

Linearly parameterized problems

$$y = Aq \,,\quad q \in \mathbb{R}^p,\ y \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times p}$$

usually $p \geq n$ $\qquad y\,| = \boxed{A}\,|\,q$

identifiable subspace $\quad I(q) = \mathcal{R}(A^T) = \mathcal{R}(A^TA)$

unidentifiable subspace $\quad NI(q) = \mathcal{N}(A) = \mathcal{N}(A^TA)$

Singular Value Decomposition (SVD)

$$A = U\Sigma V^T,\quad U \in \mathbb{R}^{n\times n},\ V \in \mathbb{R}^{p\times p}$$

orthogonal columns

$$\Sigma = \begin{bmatrix} S & 0 \end{bmatrix} \in \mathbb{R}^{n\times p}$$

$$S = \begin{pmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots\ \sigma_r & \\ 0 & & & 0 \end{pmatrix},\quad \sigma_1 \geq \sigma_2 \geq \cdots \sigma_r \geq \varepsilon$$

numerical rank r

$$U = \begin{bmatrix} U_r & U_{n-r} \end{bmatrix},\quad U_r \in \mathbb{R}^{n\times r},\ U_{n-r} \in \mathbb{R}^{n\times(n-r)}$$

$$V = \begin{bmatrix} V_r, & V_{p-r} \end{bmatrix},\quad V_r \in \mathbb{R}^{p\times r},\ V_{p-r} \in \mathbb{R}^{p\times(p-r)}$$

$$\Rightarrow A = \begin{bmatrix} U_r & U_{n-r} \end{bmatrix} \begin{bmatrix} S_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_r^T \\ V_{p-r}^T \end{bmatrix} = U_r S_r V_r^T$$

$$S_r = \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots\ \sigma_r \end{pmatrix}$$

$$v = V_{p-r}\nu \Rightarrow Av = \overbrace{AV_{p-r}}^{0}\nu = 0 \Rightarrow \mathcal{N}(A) = V_{p-r}$$

$$\nu = S_r U_r^T \mu \Rightarrow S_r^{-1}\nu = U_r^T\mu = \underbrace{U_r^T U_r}_{I}\xi \Rightarrow \mu = U_r S_r^{-1}\nu$$

$$A^T = V_r S_r U_r^T \Rightarrow v = V_r\nu = V_r S_r U_r^T \mu = A^T\mu$$

$$\Rightarrow \mathcal{R}(A^T) = V_r \Rightarrow I(q) = V_r,\ NI(q) = V_{p-r}$$

QR algorithms

$$A^T = QR = \underset{p\times n}{\square} = \underset{p\times n}{\square}\;\underset{n\times n}{\square}$$

$\mathcal{R}(A^T) = Q$ if A has full rank

$\mathcal{N}(A) = \tilde{Q} \in \mathbb{R}^{p\times(p-n)}$, $\tilde{Q}^T Q = 0$

Random algorithms

SVD, QR are applicable for small A

Stage 1: Construct low dimensional subspace when products $y = Aq$ can be computed, approximate range of A by orthonormal matrix Q of rank r such that

$$\|A - QQ^T A\|_2 \le \varepsilon \quad \begin{pmatrix}\text{if } A = QR,\ \mathcal{R}(A) = Q \\ A - QQ^T QR = A - QR = 0\end{pmatrix}$$

r as small as possible (random part)

Stage 2: Use SVD or QR and Q to approximate A (deterministic part)

$$B = Q^T A = \tilde{U}\Sigma V^T$$

$$U = Q\tilde{U} \Rightarrow QB = \underbrace{QQ^T A}_{\approx A \text{ in stage 1}} = Q\tilde{U}\Sigma V^H = U\Sigma V^H$$

$U^T U = \tilde{U}^T Q^T Q \tilde{U} = I$

Random Range Finder

1. l random q^i, compute $y^i = Aq^i \to Y = (y^1, y^2 \cdots y^l)$
2. $Y = QR$, Q has columns forming an orthonormal basis for $\mathcal{R}(Y) \Rightarrow Q$ in stage 1

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Nonlinearly parameterized problems

$y = f(q)$

local sensitivity matrix

$X_{ij} = \frac{\partial f_i}{\partial q_j}(q^*)$

linearize around q^*

local parameter identifiability in neighborhood of q^*

Exercise 6.3