Well-Quasi-Orderings
Well-Quasi-Orderings

- Quasi-Orderings
- Well-Quasi-Orderings (WQOs)
- Very-Well-Quasi-Orderings
- Building WQOs
Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

set

binary relation

quasi-order

reflexive

\[ \forall a \in A : a \sqsubseteq a \]

transitive

\[ \forall a, b, c \in A : (a \sqsubseteq b) \land (b \sqsubseteq c) \implies (a \sqsubseteq c) \]
Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

set

quasi-order

binary relation

natural numbers

\[ \langle \mathbb{N}, \leq \rangle \]

reflexive

\[ \forall a \in A : a \sqsubseteq a \]

transitive

\[ \forall a, b, c \in A : (a \sqsubseteq b) \land (b \sqsubseteq c) \Rightarrow (a \sqsubseteq c) \]

\[ 2 \leq 2 \]

\[ 2 \leq 4 \]

\[ 4 \leq 7 \]

\[ 2 \leq 7 \]
A quasi-order is a binary relation \( \preceq \) on a set \( A \) that satisfies the following properties:

- **Reflexivity**: For all \( a \in A \), \( a \preceq a \).
- **Transitivity**: For all \( a, b, c \in A \), if \( a \preceq b \) and \( b \preceq c \), then \( a \preceq c \).

This relation can also be characterized in terms of the properties:

- **Complete:** For all \( a, b \in A \), either \( a \preceq b \) or \( b \preceq a \).
- **Anti-symmetry:** For all \( a, b \in A \), if \( a \preceq b \) and \( b \preceq a \), then \( a = b \).

Examples of quasi-orders include:

- The \( \leq \) relation on the set of natural numbers \( \mathbb{N} \), which is reflexive, transitive, and complete.
- The \( \leq \) relation on the set of integers \( \mathbb{I} \), which is also reflexive, transitive, and complete.
- The \( = \) relation on the set of natural numbers \( \mathbb{N} \), which is reflexive, symmetric, and transitive.
- The \( = \) relation on the set of finite sets \( A \), which is reflexive, symmetric, and transitive.
Quasi-Ordering: \(\{a, b\} \subseteq \{a, b\}\)

**set**

\(\langle A, \subseteq \rangle\)

**quasi-order**

**binary relation**

**reflexive**

\(\forall a \in A: a \subseteq a\)

\(\forall a, b, c \in A: (a \subseteq b) \land (b \subseteq c) \implies (a \subseteq c)\)

**transitive**

\(\{a, b\} \subseteq \{a, b, c\}\)

finite sets over \(A\)

\(\langle 2^A, \subseteq \rangle\)

subset relation

\(\{a, b, c\} \subseteq \{a, b, c, d\}\)

\(\{a, b\} \subseteq \{a, b, c, d\}\)
Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

- **Set**
- **Quasi-order**
- **Binary relation**
- **Reflexive**: \( \forall a \in A : a \sqsubseteq a \)
- **Transitive**: \( \forall a, b, c \in A : (a \sqsubseteq b) \land (b \sqsubseteq c) \Rightarrow (a \sqsubseteq c) \)

**Finite multisets over** \( A \):

\[ A = \{a, b, c, d\} \]
\[ [a, a, b, c] \sqsubseteq [a, a, a, b, c, c, d] \]
Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

- **Set**: \( A \)
- **Quasi-order**: \( \sqsubseteq \)
- **Binary relation**
- **Reflexive**
  \[ \forall a \in A : a \sqsubseteq a \]
- **Transitive**
  \[ \forall a, b, c \in A : (a \sqsubseteq b) \land (b \sqsubseteq c) \implies (a \sqsubseteq c) \]

**finite multisets over** \( A \)

\[ \langle A^\bullet, \sqsubseteq \rangle \]

**Multiset ordering**

**finite set**

\( A = \{a, b, c, d\} \)

\[ [a, a, b, c] \sqsubseteq [a, a, a, b, c, c, d] \]
Quasi-Ordering

\( \langle A, \sqsubseteq \rangle \)

set

binary relation

quasi-order

reflexive

\( \forall a \in A : a \sqsubseteq a \)

transitive

\( \forall a, b, c \in A : (a \sqsubseteq b) \land (b \sqsubseteq c) \Rightarrow (a \sqsubseteq c) \)

finite multisets over \( A \)

\( \langle A^\ast, \sqsubseteq \rangle \)

finite set

multiset ordering

\( A = \{a, b, c, d\} \)

[\([a, a, b, c] \sqsubseteq [a, a, a, a, b, c, c, c, d]\)]
Quasi-Ordering

\( \langle A, \sqsubseteq \rangle \)

- set
- quasi-order
- binary relation

- finite multisets over \( A \)
- finite set
- multiset ordering

\( \forall a \in A : a \sqsubseteq a \) (reflexive)

\( \forall a, b, c \in A : (a \sqsubseteq b) \land (b \sqsubseteq c) \implies (a \sqsubseteq c) \) (transitive)

\( A = \{a, b, c, d\} \)

\( [a, a, b, c] \sqsubseteq [a, a, a, b, c, c, d] \)
Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

Set

Quasi-order

Binary relation

\[ \forall a \in A : a \sqsubseteq a \]

Reflexive

\[ \forall a, b, c \in A : (a \sqsubseteq b) \land (b \sqsubseteq c) \Rightarrow (a \sqsubseteq c) \]

Transitive

Finite set

Multiset ordering

Finite multisets over \( A \)

\[ A = \{a, b, c, d\} \]

\[ [a, a, b, c] \sqsubseteq [a, a, a, b, c, c, d] \]
Quasi-Ordering

\[ a, a, b, c, d \] \sqsubseteq \[ a, a, a, b, c, c, d \]

\[ [2,1,1,0] \sqsubseteq [3,1,2,1] \]

\[ a^2, b, c \] \sqsubseteq [a^3, b, c^2, d] \]

A = \{a, b, c, d\}

[\{a, a, b, c\} \sqsubseteq [\{a, a, a, b, c, c, d\]
Quasi-Ordering

\((A, \sqsubseteq)\) is a quasi-ordering if it is reflexive and transitive.

- Reflexive: \(\forall a \in A : a \sqsubseteq a\)
- Transitive: \(\forall a, b, c \in A : (a \sqsubseteq b) \land (b \sqsubseteq c) \Rightarrow (a \sqsubseteq c)\)

Finite sets and multisets over set \(A\):

- \(\langle A, \sqsubseteq \rangle\) is a binary relation.
- \(\langle A^\star, \sqsubseteq \rangle\) is a multiset ordering.
- \(\langle A^\ast, \sqsubseteq \rangle\) is a multiset ordering.
- \(m_1 \sqsubseteq m_2 : |m_1| \leq |m_2|\)
Well-Quasi-Orderings

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Well-Quasi-Ordering

\( \langle A, \subseteq \rangle \)

quasi-order

well-quasi-order

good sequence

infinite sequence of elements from \( A \)

\( a_0, a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots \)

\( \exists i, j : (i < j) \land (a_i \subseteq a_j) \)

WQO = all sequences are good

"... for any infinite sequence of elements in \( A \), there are two elements such that the later element is larger (wrt. \( \subseteq \)) than the earlier element ..."
Well-Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

infinite sequence of elements from \( A \)

\[ a_0, a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots \]

\[ \exists i, j : (i < j) \land (a_i \sqsubseteq a_j) \]

\[ \langle \mathbb{N}, \leq \rangle \]

WQO = all sequences are good

natural numbers

9 7 5 4 3 0 8
Well-Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

quasi-order

infinite sequence of elements from \( A \)

good sequence

\[ a_0, a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots \]

\[ \exists i, j : (i < j) \land (a_i \sqsubseteq a_j) \]

bad sequence

\[ \langle \mathbb{N}, \leq \rangle \quad \checkmark \]

\[ \langle \mathbb{I}, \leq \rangle \quad \times \]

integers

\[ 9 \quad 7 \quad 0 \quad -2 \quad -5 \quad -10 \quad -15 \quad \ldots \]
Well-Quasi-Ordering

\[ \langle A, \subseteq \rangle \]

quasi-order

infinite sequence of elements from A

good sequence

\[ a_0, a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots \]

\[ \exists i, j : (i < j) \land (a_i \subseteq a_j) \]

bad sequence

\[ \langle \mathbb{N}, \leq \rangle \]

\[ \langle \mathbb{I}, \leq \rangle \]

\[ \langle \mathbb{N}, = \rangle \]

natural numbers

[9, 7, 0, 6, 5, 10, 15, ...]
Well-Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

\text{quasi-order}

\text{infinite sequence of elements from } A

\text{good sequence}

\exists i, j : (i < j) \land (a_i \sqsubseteq a_j)

\[ a_0, a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots \]

\[ \langle \mathbb{N}, \leq \rangle \quad \checkmark \]
\[ \langle \mathbb{I}, \leq \rangle \quad \times \]
\[ \langle \mathbb{N}, = \rangle \quad \times \]

\text{finite set}

\[ a \ b \ c \ b \]

\[ A = \{a, b, c\} \]
Well-Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

quasi-order

infinite sequence of elements from \( A \)

good sequence

\[ a_0, a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots \]

\[ \exists i, j : (i < j) \land (a_i \sqsubseteq a_j) \]

finite multisets over \( A \)

finite set

multiset ordering

\[ \langle A^\times, \sqsubseteq \rangle \]
Well-Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

quasi-order

infinite sequence of elements from A

good sequence

\[ a_0, a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots \]

\[ \exists i, j : (i < j) \land (a_i \sqsubseteq a_j) \]

finite multisets over A

finite set

multiset ordering

\[ \langle A^\ast, \sqsubseteq \rangle \]

[\{a, a, b, b, b, b\} \in A^\ast]

\# a = 2

\# b = 4

A = \{a, b\}

(2, 4)
Well-Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

quasi-order

infinite sequence of elements from \( A \)

\[ \exists i, j : (i < j) \land (a_i \sqsubseteq a_j) \]

good sequence

finite multiset over \( A \)

finite set

multiset ordering

\[ \langle A^\ast, \sqsubseteq \rangle \]

\[ A = \{a, b\} \]

\[ [a, a, b, b, b, b] \in A^\ast \]

\# a = 2

\# b = 4

(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) .....
forbidden

(11,4)

(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) …..
(7,7) (11,4) (2,9) (13,2) (4,8) (0,12) (14,0) …..
Dickson's Lemma, 1910

Multiset ordering $4$ $(7,7)$ $(11,4)$ $(2,9)$ $(13,2)$ $(4,8)$ $(0,12)$ $(14,0)$ ....

$A = \{a, b\}$

$[a, a, b, b, b, b] \in A^*$

$a = 2$

$b = 4$
A well-quasi-ordering is a quasi-order \( \langle A, \sqsubseteq \rangle \) on a set \( A \) that does not contain an infinite sequence of elements \( a_0, a_1, a_2, \ldots, a_i, \ldots \) such that for all \( i < j \) it holds that \( a_i \not\sqsubseteq a_j \).

### Examples

- \( \langle \mathbb{N}, \leq \rangle \) is a well-ordering, not a well-quasi-ordering.
- \( \langle \mathbb{I}, \leq \rangle \) is a partial order, not a well-quasi-ordering.
- \( \langle \mathbb{N}, = \rangle \) is an equivalence relation, not a well-quasi-ordering.
- \( \langle A, = \rangle \) is a well-quasi-ordering if \( A \) is a finite set.
- \( \langle A^\ast, \leq \rangle \) is a well-quasi-ordering if \( A \) is a finite set and \( A^\ast \) is a finite set of multisets over \( A \) with \( \leq \) denoting multiset ordering.

Finite multisets over \( A \) with multiset ordering \( \leq \) is a well-quasi-ordering.
Well-Quasi-Orderings

- Quasi-Orderings
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- Building WQOs
Well-Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

**quasi-order**

**infinite sequence of elements from A**

\[ a_0, a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots \]

\[ \exists i, j : (i < j) \land (a_i \sqsubseteq a_j) \]

\[ \sqsubseteq \]

**good sequence**

**very good sequence**

**Very WQO = all sequences are very good**

\[ a_0, a_1, a_2, \ldots, a_{i_1}, \ldots, a_{i_2}, \ldots, a_{i_3}, \ldots \]

\[ \sqsubseteq \]

\[ \sqsubseteq \]

\[ \sqsubseteq \]

\[ \exists i_1, i_2, i_3, \ldots : (i_1 < i_2 < i_3 < \ldots) \land (a_{i_1} \sqsubseteq a_{i_2} \sqsubseteq a_{i_3} \sqsubseteq \cdots) \]
Well-Quasi-Ordering

WQO = very WQO? yes
- very WQO is WQO? yes (obvious)
- WQO is very WQO? more difficult .... yes

why?
Well-Quasi-Ordering

WQO = very WQO? yes
- very WQO is WQO? yes (obvious)
- WQO is very WQO? more difficult .... yes

\[ \forall j > i : a_i \not\subset a_j \]

terminal

\[ a_0 \ a_1 \ a_2 \ \ldots \ \ a_m \ a_{m+1} \ \ldots, \ a_n, \ \ldots \]
Well-Quasi-Ordering

WQO = very WQO? yes
- very WQO is WQO? yes (obvious)
- WQO is very WQO? more difficult .... yes

why?

∀j > i: a_i ∉ a_j

terminal

a_0 a_1 a_2 ... a_m a_{m+1} ... a_n ...
Well-Quasi-Ordering

WQO = very WQO? yes
  • very WQO is WQO? yes (obvious)
  • WQO is very WQO? more difficult .... yes

\[ \forall j > i : a_i \not\subset a_j \]

terminal

\[ a_0 \ a_1 \ a_2 \ \ldots \ \circ a_m \ a_{m+1} \ \ldots, \ \circ a_n, \ \ldots \]
WQO = very WQO? yes
• very WQO is WQO? yes (obvious)
• WQO is very WQO? more difficult yes
  • finitely many terminals why?

assume there are infinitely many terminals
Well-Quasi-Ordering

WQO = very WQO? yes
• very WQO is WQO? yes (obvious)
• WQO is very WQO? more difficult yes
  • finitely many terminals

why?

assume there are infinitely many terminals

bad sequence

$\forall j > i : a_i \not\subseteq a_j$

$a_0 \ a_1 \ a_2 \ \ldots \ a_{i_0} \ \ldots \ a_{i_1} \ \ldots \ a_{i_2} \ \ldots \ a_{i_3} \ \ldots$
Well-Quasi-Ordering

WQO = very WQO? yes
- very WQO is WQO? yes (obvious)
- WQO is very WQO? more difficult yes
  - finitely many terminals why?

assume there are infinitely many terminals

\[ \forall j > i : a_i \not\sqsubseteq a_j \]

bad sequence

\[ a_{i_0} \ a_{i_1} \]

\[ a_0 \ a_1 \ a_2 \ \ldots \ a_{i_0} \ \ldots \ a_{i_1} \ \ldots \ a_{i_2} \ \ldots \ a_{i_3} \ \ldots \]
Well-Quasi-Ordering

WQO = very WQO?  yes
  • very WQO is WQO?  yes (obvious)
  • WQO is very WQO?  more difficult  yes
    • finitely many terminals  why?

assume there are infinitely many terminals

bad sequence

\[ a_{i_0} \ a_{i_1} \ a_{i_2} \]

\[ \forall j > i : a_i \not\subseteq a_j \]

\[ a_0 \ a_1 \ a_2 \ \ldots \ a_{i_0} \ \ldots \ a_{i_1} \ \ldots \ a_{i_2} \ \ldots \ a_{i_3} \ \ldots \]
Well-Quasi-Ordering

WQO = very WQO? yes
  • very WQO is WQO? yes (obvious)
  • WQO is very WQO? more difficult yes
    • finitely many terminals

assume there are infinitely many terminals

\( \forall j > i : a_i \not\sqsubseteq a_j \)

bad sequence

\( a_{i_0} \ a_{i_1} \ a_{i_2} \ a_{i_3} \ ... \)

\( a_0 \ a_1 \ a_2 \ ... \ a_{i_0} \ ... \ a_{i_1} \ ... \ a_{i_2} \ ... \ a_{i_3} \ ... \)
WQO = very WQO? yes
- very WQO is WQO? yes (obvious)
- WQO is very WQO? more difficult .... yes
  - finitely many terminals

very good sequence

∀j > i : a_i ∉ a_j

last terminal

a_0 a_1 a_2 ... a_m ...
Well-Quasi-Ordering

WQO = very WQO? yes
  • very WQO is WQO? yes (obvious)
  • WQO is very WQO? more difficult .... yes
    • finitely many terminals

very good sequence

\[ b_0 \]

\[ \forall j > i : a_i \not\subseteq a_j \]

last terminal

\[ a_0 \ a_1 \ a_2 \ \ldots \ a_m \ \ldots \ b_0 \ \ldots \]
Well-Quasi-Ordering

\[ b_0 \sqsubseteq b_1 \]

\[ \forall j > i : a_i \nsubseteq a_j \]

very good sequence

WQO = very WQO? yes
- very WQO is WQO? yes (obvious)
- WQO is very WQO? more difficult .... yes
  - finitely many terminals

\[ a_0 \ a_1 \ a_2 \ldots \ a_m \ldots \ b_0 \ldots \ b_1 \ldots \]
Well-Quasi-Ordering

**WQO = very WQO?** yes
- very WQO is WQO? yes (obvious)
- WQO is very WQO? more difficult .... yes
  - finitely many terminals

very good sequence

\[ b_0 \subseteq b_1 \subseteq b_2 \subseteq \ldots \]

\[ \forall j > i : a_i \not\subseteq a_j \]

last terminal

\[ a_0 \ a_1 \ a_2 \ \ldots \ a_m \ \ldots \ b_0 \ \ldots \ b_1 \ \ldots \ b_2 \ \ldots \]
Well-Quasi-Ordering

WQO = very WQO? yes
- very WQO is WQO? yes (obvious)
- WQO is very WQO? more difficult .... yes

very good sequence

\[ b_0 \sqsubseteq b_1 \sqsubseteq b_2 \sqsubseteq \ldots \]

\forall j > i : a_i \not\sqsubseteq a_j

last terminal

\[
\begin{array}{cccccccc}
  a_0 & a_1 & a_2 & \ldots & a_m & \ldots & b_0 & \ldots & b_1 & \ldots & b_2 & \ldots
\end{array}
\]
Well-Quasi-Orderings

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Well-Quasi-Ordering

Higman’s Lemma

finite words over $A$

finite set

sub-word ordering

Subword Relation

$ab \sqsubseteq xaybz$

finite words

$\langle A^*, \sqsubseteq \rangle$

Higman’s Lemma

$\exists i, j : (i < j) \land (w_i \sqsubseteq w_j)$

$w_0, w_1, w_2, \ldots, w_i, \ldots, w_j, \ldots$
Well-Quasi-Ordering

Higman’s Lemma

Subword Relation

\( ab \sqsubseteq xaybz \)

“Proof”

\[
\begin{align*}
(x \sqsubseteq y) & \implies (a \cdot x \sqsubseteq a \cdot y) & (a \cdot x \not\sqsubseteq a \cdot y) & \implies (x \not\sqsubseteq y) \\
(x \sqsubseteq y) & \implies (x \sqsubseteq a \cdot y) & (x \not\sqsubseteq a \cdot y) & \implies (x \not\sqsubseteq y)
\end{align*}
\]
Well-Quasi-Ordering  Higman's Lemma

Subword Relation

\[ ab \sqsubseteq xaybz \]

"Proof"

\[ (x \sqsubseteq y) \implies (a \cdot x \sqsubseteq a \cdot y) \]

\[ (x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y) \]

\[ (a \cdot x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y) \]

\[ (x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y) \]

\[ cabc \sqsubseteq cachaca \]

\[ abc \sqsubseteq acbaca \]
Well-Quasi-Ordering

Subword Relation

Higman’s Lemma

\[ x \sqsubseteq y \implies (a \cdot x \sqsubseteq a \cdot y) \]

\[ (a \cdot x \nsubseteq a \cdot y) \implies (x \nsubseteq y) \]

\[ (x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y) \]

\[ (x \nsubseteq a \cdot y) \implies (x \nsubseteq y) \]
Well-Quasi-Ordering

Higman’s Lemma

$cabc \sqsubseteq cachbaca$

$abc \sqsubseteq acbaca$

$abc \sqsubseteq cachbaca$

$abc \not\sqsubseteq acacb$

WQO

Subword Relation

$ab \sqsubseteq xaybz$

“Proof”

$(x \sqsubseteq y) \implies (a \cdot x \sqsubseteq a \cdot y)$

$(a \cdot x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$

$(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y)$

$(x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$

$(x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y)$
Well-Quasi-Ordering

Higman’s Lemma

Subword Relation

\[ ab \sqsubseteq xaybz \]

“Proof”

\[
\begin{align*}
abc & \sqsubseteq cachaca \\
abc & \sqsubseteq cachaca \\
abc & \sqsubseteq cachaca \\
bc & \not\sqsubseteq cachb \\
abc & \not\sqsubseteq acachb \\
abc & \not\sqsubseteq cachb
\end{align*}
\]

\[
\begin{align*}
(x \sqsubseteq y) & \implies (a \cdot x \sqsubseteq a \cdot y) \\
(x \sqsubseteq y) & \implies (x \sqsubseteq a \cdot y) \\
(a \cdot x \not\sqsubseteq a \cdot y) & \implies (x \not\sqsubseteq y) \\
(x \not\sqsubseteq a \cdot y) & \implies (x \not\sqsubseteq y)
\end{align*}
\]
Well-Quasi-Ordering

"minimal" bad sequence:

\[ w_1 \]

: a shortest word starting a bad sequence

Higman's Lemma

Subword Relation

Proof

\[ \text{ab} \sqsubseteq \text{xayb} \]

WQO
Well-Quasi-Ordering

“minimal” bad sequence:

\( w_1 \) : a shortest word starting a bad sequence
\( w_2 \) : a shortest word \( v \) such that \( w_1 v \cdots \) is bad

Subword Relation

\( ab \sqsubseteq xaybz \)

\( w_1 \quad w_2 \quad \cdots \)
Well-Quasi-Ordering

"minimal" bad sequence:

\( w_1 \) : a shortest word starting a bad sequence
\( w_2 \) : a shortest word \( v \) such that \( w_1v \cdots \) is bad
\( w_3 \) : a shortest word \( v \) such that \( w_1w_2v \cdots \) is bad

Subword Relation

\[ ab \sqsubseteq xaybzx \]

WQO

minimal bad sequence

\( w_1 \ w_2 \ w_3 \ \cdots \)
"minimal" bad sequence:

\( w_1 : \) a shortest word starting a bad sequence

\( w_2 : \) a shortest word \( v \) such that \( w_1 v \cdots \) is bad

\( w_3 : \) a shortest word \( v \) such that \( w_1 w_2 v \cdots \) is bad

\( w_n : \) a shortest word \( v \) such that \( w_1 w_2 w_3 \cdots w_{n-1} v \) is bad

\( w_1 \ w_2 \ w_3 \ \cdots \ w_n \)
Well-Quasi-Ordering

“minimal” bad sequence:

- $w_1$: a shortest word starting a bad sequence
- $w_2$: a shortest word $v$ such that $w_1v\cdots$ is bad
- $w_3$: a shortest word $v$ such that $w_1w_2v\cdots$ is bad
- $w_n$: a shortest word $v$ such that $w_1w_2w_3\cdots w_{n-1}v$ is bad

Infinite many start with some “a”

$w_1 \ w_2 \ w_3 \ \cdots \ w_n \ a \cdot v_{i_1} \ \cdots \ a \cdot v_{i_2} \ \cdots \ a \cdot v_{i_3} \ \cdots$

Subword Relation

$ab \sqsubseteq xayb$
Well-Quasi-Ordering

Subword Relation

Well-Quasi-Ordering

“minimal” bad sequence:

\[ w_1 : \text{a shortest word starting a bad sequence} \]
\[ w_2 : \text{a shortest word } v \text{ such that } w_1 v \cdots \text{ is bad} \]
\[ w_3 : \text{a shortest word } v \text{ such that } w_1 w_2 v \cdots \text{ is bad} \]
\[ w_n : \text{a shortest word } v \text{ such that } w_1 w_2 w_3 \cdots w_{n-1} v \text{ is bad} \]
Well-Quasi-Ordering

“minimal” bad sequence:

$w_1$ : a shortest word starting a bad sequence

$w_n$ : a shortest word $w_n$ such that $w_n$ is a bad sequence

Subword Relation

\[ (x ⊑ y) \implies (a \cdot x \subseteq a \cdot y) \]
\[ (a \cdot x \not\subseteq a \cdot y) \implies (x \not\subseteq y) \]

\[ (x \subseteq y) \implies (x \subseteq a \cdot y) \]
\[ (x \not\subseteq a \cdot y) \implies (x \not\subseteq y) \]

Well-Quasi-Ordering

Higman's Lemma
Well-Quasi-Ordering

“minimal” bad sequence:

\( w_1 \): a shortest word starting a bad sequence

\( w_2 \): a shortest word \( w \) such that \( w_1 \cdot w \) is bad

\( w_3 \): a shortest word \( w \) such that \( w_2 \cdot w \) is bad

\( \vdash \): a shortest word \( w \) such that \( w_n \cdot w \) is bad

Subword Relation

\( ab \sqsubseteq xaybz \)

WQO
Well-Quasi-Ordering

“minimal” bad sequence:

$w_1$: a shortest word starting a bad sequence

Well-a shortest word $w$ such that $w_1 w_2 w_3 \ldots w_n$ is bad.

\[ (x \sqsubseteq y) \implies (a \cdot x \sqsubseteq a \cdot y) \quad (a \cdot x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y) \]

\[ (x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y) \quad (x \not\sqsubseteq a \cdot y) \implies (x \not\sqsubseteq y) \]

Subword Relation

Higman’s Lemma

\[ ab \sqsubseteq xaybzx \]

WQO
Well-Quasi-Ordering

“minimal” bad sequence:

\( w_1 \): a shortest word starting a bad sequence
\( w_2 \): a shortest word \( \nu \) such that \( w_1 \nu \cdot \cdot \cdot \) is bad
\( w_3 \): a shortest word \( \nu \) such that \( w_1 w_2 \nu \cdot \cdot \cdot \) is bad
\( w_n \): a shortest word \( \nu \) such that \( w_1 w_2 w_3 \cdot \cdot \cdot w_{n-1} \nu \) is bad

Higman’s Lemma

Subword Relation

\[ ab \sqsubseteq xayb\]
Well-Quasi-Ordering

Higman’s Lemma

“minimal” bad sequence:

\(w_1\) : a shortest word starting a bad sequence
\(w_2\) : a shortest word \(v\) such that \(w_1v\ldots\) is bad
\(w_3\) : a shortest word \(v\) such that \(w_1w_2v\ldots\) is bad
\(w_n\) : a shortest word \(v\) such that \(w_1w_2w_3\ldots w_{n-1}v\) is bad

Subword Relation

\(ab \sqsubseteq xaybz\)

WQO

minimal bad sequence

infinite sequence

infinitely many start with some “a”

bad sequence

contradiction
Well-Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \quad \text{wqo} \quad \langle A^*, \sqsubseteq^* \rangle \quad \text{wqo?} \]

Set of finite words over \( A \)

\[ a_1 a_2 \cdots a_n \sqsubseteq^* x_0 b_1 x_1 b_2 x_2 \cdots x_{n-1} b_n x_n \]

if

\[ a_1 \sqsubseteq b_1 \quad a_2 \sqsubseteq b_2 \quad \cdots \quad a_n \sqsubseteq b_n \]

\[ \langle \mathbb{N}, \leq \rangle \quad \langle \mathbb{N}^*, \leq^* \rangle \]

\[ 175 \leq^* 0329368 \]
“minimal” bad sequence:

\[ w_1 : \text{a shortest word starting a bad sequence} \]
“minimal” bad sequence:

\( w_1 \) : a shortest word starting a bad sequence

\( w_2 \) : a shortest word \( v \) such that \( w_1 v \cdots \) is bad
“minimal” bad sequence:

\( w_1 \) : a shortest word starting a bad sequence

\( w_2 \) : a shortest word \( v \) such that \( w_1 v \cdots \) is bad

\( w_3 \) : a shortest word \( v \) such that \( w_1 w_2 v \cdots \) is bad

\[ w_1 \quad w_2 \quad w_3 \quad \cdots \]
“minimal” bad sequence:

- $w_1$: a shortest word starting a bad sequence
- $w_2$: a shortest word $v$ such that $w_1v\cdots$ is bad
- $w_3$: a shortest word $v$ such that $w_1w_2v\cdots$ is bad
- $w_n$: a shortest word $v$ such that $w_1w_2w_3\cdots w_{n-1}v$ is bad

$w_1\ w_2\ w_3\ \cdots\ \cdots\ w_n\ \cdots$
Well-Quasi-Ordering

Higman’s Lemma

“minimal” bad sequence:

\( w_1 \): a shortest word starting a bad sequence
\( w_2 \): a shortest word \( v \) such that \( w_1 v \cdots \) is bad
\( w_3 \): a shortest word \( v \) such that \( w_1 w_2 v \cdots \) is bad
\( w_n \): a shortest word \( v \) such that \( w_1 w_2 w_3 \cdots w_{n-1} v \) is bad

Minimal bad sequence → Infinite sequence → Very WQO
“minimal” bad sequence:

- $w_1$ : a shortest word starting a bad sequence
- $v_1$ : a shortest word such that $w_1 \cdot v_1$ is bad
- $w_2$ : a shortest word such that $w_2 \cdot v_2$ is bad
- $w_3$ : a shortest word such that $w_3 \cdot v_3$ is bad

Well-Quasi-Ordering (WQO):

\[
(x \sqsubseteq y) \land (a \sqsubseteq b) \implies (a \cdot x \sqsubseteq b \cdot y)
\]

\[
(x \sqsubseteq y) \implies (x \sqsubseteq a \cdot y)
\]
“minimal” bad sequence:

\[ w_1 : \text{a shortest word starting a bad sequence} \]
\[ w_2 : \text{a shortest word } v \text{ such that } w_1v\cdots \text{ is bad} \]
\[ w_3 : \text{a shortest word } v \text{ such that } w_1w_2v\cdots \text{ is bad} \]
\[ w_n : \text{a shortest word } v \text{ such that } w_1w_2w_3\cdots w_{n-1}v \text{ is bad} \]
Well-Quasi-Ordering

“minimal” bad sequence:

\(w_1\) : a shortest word starting a bad sequence

\(w_2\) : a shortest word \(v\) such that \(w_1v\cdots\) is bad

\(w_3\) : a shortest word \(v\) such that \(w_1w_2v\cdots\) is bad

\(w_n\) : a shortest word \(v\) such that \(w_1w_2w_3\cdots w_{n-1}v\) is bad

Higman’s Lemma

\(a_1 \sqsubseteq a_2 \sqsubseteq a_3 \sqsubseteq \cdots\)

contradiction
Well-Quasi-Ordering

\[ \langle A, \sqsubseteq \rangle \]

WQO

\[ \langle A^*, \sqsubseteq^* \rangle \]

\[ \langle A^\otimes, \sqsubseteq^\otimes \rangle \]

set of finite multisets over \( A \)
Well-Quasi-Ordering

\[(A, \sqsubseteq)\]

\[\langle A, \sqsubseteq \rangle\]

\[\langle A^*, \sqsubseteq^* \rangle\]

\[\langle A^\otimes, \sqsubseteq^\otimes \rangle\]

\[\langle A^k, \sqsubseteq^k \rangle\]

set of vectors of length \(k\) over \(A\)

\[\langle A_1 \times A_2 \times \cdots \times A_k, \sqsubseteq^k \rangle\]
Well-Quasi-Ordering

\( \langle A, \subseteq \rangle \)

wqo

\( \langle A^*, \subseteq^* \rangle \) ✓

\( \langle A^\ast, \subseteq^\ast \rangle \) ✓

\( \langle A^k, \subseteq^k \rangle \) ✓

\( \langle 2^A, \subseteq^{2A} \rangle \) ✓

set of finite sets over \( A \)
Well-Quasi-Ordering

\[ \langle A^*, \subseteq^* \rangle \]

\[ \langle A^\otimes, \subseteq^\otimes \rangle \]

\[ \langle A^k, \subseteq^k \rangle \]

\[ \langle 2^A, \subseteq 2^A \rangle \]

\[ \langle A, \subseteq \rangle \]

\[ \text{wqo} \]

set of finite words over \( A \)

set of finite multisets over \( A \)

set of vectors of length \( k \) over \( A \)

set of finite sets over \( A \)
Well-Quasi-Ordering

- \(\langle A^*, \subseteq^* \rangle\)
- \(\langle A^\otimes, \subseteq^\otimes \rangle\)
- \(\langle A^k, \subseteq^k \rangle\)
- \(\langle 2^A \subseteq 2^A \rangle\)

natural numbers: \(\langle \mathbb{N}, \leq \rangle\)

finite set: \(\langle A, = \rangle\)

standard ordering

subword relation
Well-Quasi-Ordering

\( \langle A^*, \leq^* \rangle \)
\( \langle A^\otimes, \leq^\otimes \rangle \)
\( \langle A^k, \leq^k \rangle \)
\( \langle 2^A \subseteq 2^A \rangle \)

\( \langle \mathbb{N}, \leq \rangle \)
\( \langle A, = \rangle \)

\( \langle \mathbb{N}^\otimes, \leq^\otimes \rangle \)

Graph with nodes labeled 2, 3, 5, 7, 1, 6, 4 connected with arrows.
Well-Quasi-Ordering

\[ \langle \mathbb{N}^\ast, (\leq^\ast)^\ast \rangle \]

\[ \langle \mathbb{N}, \leq \rangle \]

\[ \langle A, = \rangle \]

\[ \langle A^\ast, \sqsubseteq^\ast \rangle \]

\[ \langle A^\ast, \sqsubseteq^\ast \rangle \]

\[ \langle A^k, \sqsubseteq^k \rangle \]

\[ \langle 2^A \sqsubseteq 2^A \rangle \]
Well-Quasi-Orderings

- Quasi-Orderings
- Well-Quasi-Orderings (WQOs)
- Very-Well-Quasi-Orderings
- Building WQOs