Math308, Quiz 9, 4/04/14

First Name: .... **Last Name: ....**

Table 1: Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1	$\frac{1}{s} \quad s > 0$	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}} \quad s > 0$
$e^{-\alpha t}$	$\frac{1}{s+\alpha} \quad s > -\alpha$	$e^{-\alpha t} t^n$	$\frac{n!}{(s+\alpha)^{n+1}} \quad s > -\alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2} \quad s > 0$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2} \quad s > 0$
$e^{\alpha t} \sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2} \quad s > \alpha$	$e^{\alpha t} \cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2} \quad s > 0$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2} \quad s > \lvert\omega\rvert$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2} \quad s > \lvert\omega\rvert$
$u_\alpha(t)$	$\frac{e^{-\alpha s}}{s} \quad s > 0$	$\delta(t-\alpha)$	$e^{-\alpha s} \quad s > -\infty$

Theorem. Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose that there exist constants K, a and M such that $|f(t)| \le Ke^{at}, |f'(t)| \le Ke^{at}, \ldots, |f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}[f^{(n)}(t)]$ exists for $s > a$ and given by

- $\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$.
- if $F(s) = \mathcal{L}[f(t)]$ for $s > a$, then $\mathcal{L}[u_c(t) f(t-c)] = e^{-cs}\mathcal{L}[f(t)] = e^{-cs} F(s)$ for $s > a, c > 0$.
- if $f(t) = \mathcal{L}^{-1}[F(s)]$, then $u_c(t) f(t-c) = \mathcal{L}^{-1}[e^{-cs} F(s)]$.

Grade: ....

Show all work!

Problem 1. 50%. Find the Laplace transform of the following function

$$f(t) = \begin{cases} 0, & t < 10, \\ (t-10)^4, & t \geq 10. \end{cases}$$

Problem 2. 50%. Find the inverse Laplace transform of the following function

$$F(s) = \frac{1 - e^{-5s}}{s^4}.$$

Solutions

Problem 1. 50%. First we rewrite the given function in terms of Heaviside function:

$$f(t) = \left\{ \begin{matrix} 0, & t < 10, \\ (t-10)^4, & t \geq 10. \end{matrix} \right\} = u_{10}(t)(t-10)^4.$$

By using the above theorem and the table, we get

$$\mathcal{L}[f(t)] = \mathcal{L}[u_{10}(t)(t-10)^4] = e^{-10s}\mathcal{L}[t^4] = 4!\,\frac{e^{-10s}}{s^5} = 24\frac{e^{-10s}}{s^5}.$$

Problem 2. 50%. We get

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1-e^{-5s}}{s^4}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^4}\right] - \mathcal{L}^{-1}\left[\frac{e^{-5s}}{s^4}\right] = \frac{1}{3!}t^3 - \mathcal{L}^{-1}\left[e^{-5s}F_1(s)\right],$$

where we denote $F_1(s) = \frac{1}{s^4}$. Using the above theorem if $f_1(t) = \mathcal{L}^{-1}[F_1(s)]$ then

$$\mathcal{L}^{-1}[e^{-5s}F_1(s)] = u_5(t)f(t-5),$$

and therefore $f_1(t) = \mathcal{L}^{-1}[\frac{1}{s^4}] = \frac{1}{3!}t^3$ and

$$\mathcal{L}^{-1}\left[\frac{e^{-5s}}{s^4}\right] = \frac{1}{6}u_5(t)(t-5)^3.$$

Finally we get

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{6}\left(t^3 - u_5(t)(t-5)^3\right).$$