Math308, Quiz 2, 01/31/14

First Name: ....   **Last Name:** ....

Grade: ....

Show all work!

Consider the following initial value problem:

$$\begin{aligned} & y' + \frac{1}{t} y = 3t^2 + 2t, \quad t > 0, \\ & y(1) = 2. \end{aligned} \tag{1}$$

Problem 1. 20%. Without solving the problem, show that (1) has a unique solution.

Problem 2. 80%. Solve the problem (1).

Solutions

Problem 1. It is easy to see that functions $p(t) = \frac{1}{t}$ and $g(t) = 3t^2 + 2t$ are continuous for $t > 0$, and $t_0 = 1 \in (0, \infty)$. Therefore according to Theorem 2.4.1 of the book (1) has a unique solution.

Problem 2. Multiply the equation by a function $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + \mu(t)\frac{1}{t}y = \mu(t)(3t^2 + 2t). \tag{2}$$

The left hand side of the last equation is identical to $\frac{d}{dt}(\mu(t)y)$ if

$$\frac{d\mu(t)}{dt} = \mu(t)\frac{1}{t},$$

which can be easily solved for $\mu(t)$:

$$\frac{d\mu(t)}{\mu(t)} = \frac{1}{t}dt \quad \Rightarrow \ln|\mu(t)| = \ln|t| + C \quad \Rightarrow \mu(t) = Ct.$$

Set $C = 1$ for simplicity, then (2) becomes

$$t\frac{dy}{dt} + y = t(3t^2 + 2t),$$

or

$$\frac{d}{dt}(ty) = 3t^3 + 2t^2.$$

By integration we obtain:

$$ty = \frac{3}{4}t^4 + \frac{2}{3}t^3 + C \quad \Rightarrow y = \frac{3}{4}t^3 + \frac{2}{3}t^2 + \frac{C}{t}.$$

Now, using the initial data $y(1) = 2$ we find the coefficient C:

$$2 = \frac{3}{4} + \frac{2}{3} + C \quad \Rightarrow C = \frac{7}{12}.$$

Thus the desired solution to the initial value problem is

$$y = \frac{3}{4}t^3 + \frac{2}{3}t^2 + \frac{7}{12t}.$$