Math308, Quiz 10, 4/07/14

First Name: .... **Last Name: ....**

Table 1: Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1	$\frac{1}{s} \quad s > 0$	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}} \quad s > 0$
$e^{-\alpha t}$	$\frac{1}{s+\alpha} \quad s > -\alpha$	$e^{-\alpha t} t^n$	$\frac{n!}{(s+\alpha)^{n+1}} \quad s > -\alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2} \quad s > 0$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2} \quad s > 0$
$e^{\alpha t} \sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2} \quad s > \alpha$	$e^{\alpha t} \cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2} \quad s > 0$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2} \quad s > \lvert\omega\rvert$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2} \quad s > \lvert\omega\rvert$
$u_\alpha(t)$	$\frac{e^{-\alpha s}}{s} \quad s > 0$	$\delta(t-\alpha)$	$e^{-\alpha s} \quad s > -\infty$

Theorem. Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \leq t \leq A$. Suppose that there exist constants K, a and M such that $|f(t)| \leq Ke^{at}, |f'(t)| \leq Ke^{at}, \ldots, |f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq M$. Then $\mathcal{L}[f^{(n)}(t)]$ exists for $s > a$ and given by

- $\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$.
- if $F(s) = \mathcal{L}[f(t)]$ for $s > a$, then $\mathcal{L}[u_c(t) f(t-c)] = e^{-cs} \mathcal{L}[f(t)] = e^{-cs} F(s)$ for $s > a, c > 0$.
- if $f(t) = \mathcal{L}^{-1}[F(s)]$, then $u_c(t) f(t-c) = \mathcal{L}^{-1}[e^{-cs} F(s)]$.

Grade: ....

Show all work!

Problem 1. 100%. Find the solution of the given initial value problem

$$y'' + 4y = \delta(t - 4\pi) - 2\delta(t - \pi) + u_\pi(t), \quad y(0) = \frac{1}{2},\, y'(0) = 0.$$

Solutions

Problem 1. 100%. By taking the Laplace transform of the equation, using the above table together with initial conditions we obtain

$$\begin{aligned}
\mathcal{L}[y''+4y] &= \mathcal{L}[\delta(t-4\pi)-2\delta(t-\pi)+u_\pi(t)],\\
s^2\mathcal{L}[y]-sy(0)-y'(0)+4\mathcal{L}[y] &= \mathcal{L}[\delta(t-4\pi)]-2\mathcal{L}[\delta(t-\pi)]+\mathcal{L}[u_\pi(t)],\\
(s^2+4)\mathcal{L}[y]-\frac{1}{2}s &= e^{-4\pi s}-2e^{-\pi s}+\frac{e^{-\pi s}}{s},\\
\mathcal{L}[y] &= \frac{1}{2}\frac{s}{s^2+4}+\frac{e^{-4\pi s}-2e^{-\pi s}}{s^2+4}+\frac{e^{-\pi s}}{s(s^2+4)}.
\end{aligned}$$

The next step is taking the inverse of the Laplace transform from the last equality in order to find y. Let us calculate the inverse of the Laplace transform of the right hand side separately:

- $$\mathcal{L}^{-1}\left[\frac{1}{2}\frac{s}{s^2+4}\right]=\frac{1}{2}\mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right]=\frac{1}{2}\cos 2t.$$

- $$\mathcal{L}^{-1}\left[\frac{e^{-4\pi s}-2e^{-\pi s}}{s^2+4}\right]=\frac{1}{2}\mathcal{L}^{-1}\left[e^{-4\pi s}\frac{2}{s^2+2^2}\right]-\mathcal{L}^{-1}\left[e^{-\pi s}\frac{2}{s^2+2^2}\right]=$$
$$=\frac{1}{2}u_{4\pi}(t)\sin 2(t-4\pi)-u_\pi(t)\sin 2(t-\pi).$$

- The partial fraction gives us:
$$\frac{1}{s(s^2+4)}=\frac{A}{s}+\frac{Bs}{s^2+4}=\frac{(A+B)s^2+4A}{s(s^2+4)}\Rightarrow\begin{cases}A+B=0,\\4A=1.\end{cases}\Rightarrow\begin{cases}B=-\frac{1}{4},\\A=\frac{1}{4}.\end{cases}$$
Therefore,
$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s(s^2+4)}\right]=\mathcal{L}^{-1}\left[e^{-\pi s}\frac{1}{4}\frac{1}{s}-e^{-\pi s}\frac{1}{4}\frac{s}{s^2+4}\right]=\frac{1}{4}u_\pi(t)-\frac{1}{4}u_\pi(t)\cos(2(t-\pi))$$
$$=\frac{1}{4}u_\pi(t)(1-\cos(2(t-\pi))).$$

We now collect all terms and obtain the solution of the given initial value problem:

$$y(t)=\frac{1}{2}\cos 2t+\frac{1}{2}u_{4\pi}(t)\sin 2(t-4\pi)-u_\pi(t)\sin 2(t-\pi)+\frac{1}{4}u_\pi(t)(1-\cos(2(t-\pi))).$$

We can use the trigonometric identities $\sin(t-2n\pi)=\sin t$ and $\cos(t-2n\pi)=\cos t$ for $n=0,1,2,\ldots$ to simplify the solution:

$$y(t)=\frac{1}{4}u_\pi(t)+\left(\frac{1}{2}-\frac{1}{4}u_\pi(t)\right)\cos 2t+\left(\frac{1}{2}u_{4\pi}(t)-u_\pi(t)\right)\sin 2t.$$