Block 2x2 matrices

$$\rightarrow \begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix} \quad \begin{matrix} A(n,n) \\ B^T(n,m) \end{matrix} \qquad (1)$$

$$\rightarrow \begin{bmatrix} A & B^T \\ C & -D \end{bmatrix} \quad D(m,m) \qquad (2)$$

$$\mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad (3)$$

$$\mathcal{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & \underset{\uparrow}{S} \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1} A_{12} \\ 0 & I_2 \end{bmatrix} \qquad (4)$$

Schur complement

$$S = A_{22} - A_{21} A_{11}^{-1} A_{12}$$

Case (1) : $S = -C A^{-1} B^T$

(5)

$$\underline{\underline{\mathcal{A}}} = \begin{bmatrix} I_1 & 0 \\ A_{21} A_{11}^{-1} & I_2 \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1} A_{12} \\ 0 & I_2 \end{bmatrix}$$

$$P_D = \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}$$

$$P_T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & S \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} A_{11} & 0 \\ A_{21} & S \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$(A - P_T) \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{(\lambda - 1)}_{\mu} P_T \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} - S \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mu P_T \begin{bmatrix} x \\ y \end{bmatrix}$$

* Another approach:

$$P_T^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -S^{-1} A_{21} A_{11}^{-1} & S^{-1} \end{bmatrix}$$

$$\underbrace{P_T^{-1} A}_{} =$$

$$= \begin{bmatrix} I_1 & A_{11}^{-1} A_{12} \\ 0 & I_2 \end{bmatrix}$$

all eigenvalues = 1

P_T – the ideal preconditioner

Problems with

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & S \end{bmatrix} ?$$

$$S = A_{22} - A_{21} \underline{A_{11}^{-1}} A_{12}$$

Solve with A_{11}^{-1} within ①

In order to recover the action of S on a vector, it is required to solve systems with A_{11} up to machine accuracy!

Why? Because otherwise we perturbe S very much.

$\Rightarrow$ Too expensive!

Consider the case with approximate A_{11} and $S \to \tilde{A}_{11}, \tilde{S}$

$$\tilde{P}_T = \begin{bmatrix} \tilde{A}_{11} & 0 \\ A_{21} & \tilde{S} \end{bmatrix}.$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} \tilde{A}_{11} & 0 \\ A_{21} & \tilde{S} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Generalized eigenvalue problem. Equivalent to

$$\begin{bmatrix} \tilde{A}_{11}^{-1} & 0 \\ -\tilde{S}^{-1} A_{21} \tilde{A}_{11}^{-1} & \tilde{S}^{-1} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} \tilde{A}_{11}^{-1} A_{11} & \tilde{A}^{-1} A_{12} \\ -\tilde{S}^{-1} A_{21} \tilde{A}_{11}^{-1} A_{11} + \tilde{S}^{-1} A_{21} & * \end{bmatrix}$$

$$* = -\tilde{S}^{-1} A_{21} \tilde{A}_{11}^{-1} A_{12} + \tilde{S}^{-1} A_{22}$$

$$= \tilde{S}^{-1}\left[A_{22} - A_{21} \tilde{A}_{11}^{-1} A_{12} \pm A_{21} A_{11}^{-1} A_{12}\right]$$

$$= \tilde{S}^{-1}\left[S + A_{21}(A_{11}^{-1} - \tilde{A}_{11}^{-1}) A_{12}\right]$$

$$\tilde{S}^{-1} A_{21}(\tilde{A}_{11}^{-1} A_{11} - I_1)$$

Result:

$$\begin{bmatrix} \underline{\tilde{A}_{11}^{-1} A_{12}} & \cdots \\ \underline{\tilde{S}^{-1} A_{21}\left(I - \hat{A}_{11}^{-1} A_{11}^{-1}\right)} & \tilde{S}^{-1} S + \tilde{S}^{-1} \left(A_{21} \tilde{A}_{11}^{-1}\left(I - A_{11} \tilde{A}_{11}^{-1}\right) A_{11}\right) \end{bmatrix}$$

Moral: the approx. of A_{11} is very important!

Both $\tilde{A}_{11}$ and $\hat{S}$ must be good approx. of the exact blocks.

If $\tilde{A}_{11} = A_{11}$, then

$$\tilde{P}_{r}^{-1} A = \begin{bmatrix} \underline{I} & \cdots \\ \underline{0} & \tilde{S}^{-1} S \end{bmatrix}.$$

What about $P_D^{-1}\mathcal{A}$?

Exact A_{11}, S.

Problem is:

$$\begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \overset{?}{\lambda}$$

The classical result is for

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix}$$

$$S = CA^{-1}B^T$$

$$P_D = \begin{bmatrix} A & 0 \\ 0 & CA^{-1}B^T \end{bmatrix}$$

$$T = P_D^{-1}\mathcal{A} = \begin{bmatrix} I & A^{-1}B^T \\ (CA^{-1}B^T)C & 0 \end{bmatrix}$$

? $\lambda(P_D^{-1}\mathcal{A}) = $?

Observe:

$$\left(T-\frac{1}{2}I\right)^2 = \begin{bmatrix} \frac{1}{4}I + A^{-1}B^T(CA^{-1}B^T)^{-1}C & 0 \\ 0 & \frac{5}{4}I \end{bmatrix}$$

The matrix

$Q = A^{-1}B(CA^{-1}B^T)^{-1}C$ is

projector: $Q^2 = Q$

Then

$$\left[\left(T-\frac{1}{2}I\right)^2 - \frac{1}{4}I\right]^2 = \left[\left(T-\frac{1}{2}I\right)^2 - \frac{1}{4}I\right]$$

open brackets

Then: $T(T-I)(T^2-T-I)=0$

The characteristic polynomial of T is

$$t(t-1)(t^2-t-1)=0.$$

$t=0$ is a trivial root

$$t_1 = 1, \quad t_{2,3} = \frac{1 \pm \sqrt{4+1}}{2}.$$

The task in many cases is to find a good approx. of the Schur complement.

Examples:

Stokes problem

$$\left.\begin{aligned} -\Delta u + \nabla p &= f \\ \Delta u &= 0 \end{aligned}\right|$$

Discretize with FEM:

$$\mathcal{A} = \begin{array}{c} \left[\begin{array}{ccc|c} A_{11} & & & \\ & A_{22} & & B^T \\ & & A_{33} & \\ \hline & B & & 0 \end{array}\right] \end{array} \begin{array}{l} \}\, n \\ \\ \}\, m \end{array}$$

$$\mathcal{S} \approx M(m, m)$$

$\rightarrow$ mass matrix corresp. to p.

$$P = \begin{bmatrix} A_{11} & 0 \\ B & M \end{bmatrix}$$

A_{11} - use MG
AMG

M - use Cheb.

Navier-Stokes:
splitting techniques

1) $u^{k+1} = p^{k}$

2) p^{k+1} : $-\Delta p^{k+1} = g(u^{k+1})$

Problem parameters

{ Reynolds number,
very large