Uppsala University
Department of Information Technology
Division of Scientific Computing

Scientific Computing for Partial Differential Equations, 5.0 credits, 2023-03-15

Exam time: $8^{00} - 13^{00}$ (5 hrs)
Aids: Attached formulae, calculator, Physics Handbook, Mathematics Handbook

Unless otherwise specified, all solutions must include detailed reasoning and complete calculations.

Grade requirements:
Grade 3: At least 12/24 points.
Grade 4: At least 17/24 points.
Grade 5: At least 21/24 points.

1. Consider the following PDE and initial condition:

$$\begin{aligned} C\mathbf{u}_t &= B\mathbf{u}_x + \mathbf{F}, \quad x \in (0, L), \quad t > 0, \\ \mathbf{u} &= \mathbf{f}, \quad x \in [0, L], \quad t = 0, \end{aligned} \tag{1}$$

where $\mathbf{F} = \mathbf{F}(x, t)$ is a forcing function, $\mathbf{f} = \mathbf{f}(x)$ is initial data, and

$$C(x) = \begin{bmatrix} c_1(x) & \\ & c_2(x) \end{bmatrix}, \quad B = \begin{bmatrix} & i \\ -i & \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $c_{1,2}$ are real-valued, strictly positive functions.

(a) Use the energy method to prove that (1) is well-posed with *periodic* boundary conditions. That is, you may disregard any boundary terms. (2p)

(b) For the non-periodic initial-boundary-value problem (IBVP) to be well posed, how many boundary conditions are required at $x = 0$ and at $x = L$? Motivate your answer. (1p)

(c) Derive a set of well-posed boundary conditions. (1p)

(d) State an SBP-SAT discretization of the PDE combined with the set of boundary conditions that you derived in the previous problem. You may here assume that $\mathbf{F} = 0$ and c_1 and c_2 are *constant*. You do *not* need to prove stability. You may use up to 4 unspecified *scalar* penalty parameters in your SATs. (2p)

(a) Observations: The PDE is first-order in time so we expect *one* initial condition, which is what we have. Further, C is independent of t and

$$C = C^* > 0, \quad B = B^*.$$

Since $C = C^* > 0$, C defines an inner product and norm. For well-posedness it is sufficient to study the homogeneous problem, so we set $\mathbf{F} = 0$. We proceed with the energy method. Multiplying by $\mathbf{u}^*$ and integrating yields

$$(\mathbf{u}, \mathbf{u}_t)_C = (\mathbf{u}, B\mathbf{u}_x) = \mathbf{u}^* B\mathbf{u}|_0^L - (\mathbf{u}_x, B\mathbf{u}).$$

Taking the complex conjugate yields

$$(\mathbf{u}_t, \mathbf{u})_C = (B\mathbf{u}_x, \mathbf{u}) = (\mathbf{u}_x, B^*\mathbf{u}) = (\mathbf{u}_x, B\mathbf{u}).$$

Adding the two relations above leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_C^2 = \mathbf{u}^* B\mathbf{u}|_0^L.$$

For the periodic problem, the boundary terms vanish and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_C^2 = 0,$$

which shows that $\|\mathbf{u}\|_C^2$ does not increase with t, and thus the periodic problem is well posed.

(b) The number of BC required at each boundary depends on the eigenvalues of B. The eigenvalues are given by

$$\lambda^2 + i^2 = 0 \Leftrightarrow \lambda = \pm 1.$$

For every positive eigenvalue we need one BC at the right boundary and for every negative eigenvalue we need one BC at the left boundary. Here, we need one BC at each boundary.

Alternatively, you can argue that the boundary terms are such that one BC at each boundary is required to bound the growth of the solution. See the solution to (c).

(c) The boundary terms are

$$\mathbf{u}^* B \mathbf{u}\big|_0^L = u_1^* i u_2 - u_2^* i u_1\big|_0^L .$$

We can ensure that the boundary terms corresponding to the left boundary vanish by setting either $u_1 = 0$ or $u_2 = 0$. The same holds for the right boundary. This leads to the following 4 combinations (there are other possibilities too):

$$\begin{cases} u_1 = 0, & x = 0 \\ u_1 = 0, & x = L \end{cases} \qquad \begin{cases} u_1 = 0, & x = 0 \\ u_2 = 0, & x = L \end{cases}$$

$$\begin{cases} u_2 = 0, & x = 0 \\ u_1 = 0, & x = L \end{cases} \qquad \begin{cases} u_2 = 0, & x = 0 \\ u_2 = 0, & x = L \end{cases}$$

These sets all lead to energy conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_C^2 = 0,$$

and thus well-posedness.

(d) Let us choose the boundary conditions

$$\begin{cases} u_1 = 0, & x = 0 \\ u_1 = 0, & x = L \end{cases}.$$

Define a grid of $m+1$ points with grid spacing $h = \frac{L}{m}$:

$$x_j = jh, \quad j = 0, 1 \ldots, m.$$

Let D_1 be a corresponding SBP operator for the first derivative (see attached formulae sheet).

Alternative 1:

On component form, the PDE reads

$$
\begin{aligned}
c_1 u_{1,t} &= i u_{2,x} \\
c_2 u_{2,t} &= -i u_{1,x}
\end{aligned}
$$

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the two discrete solution vectors that approximate u_1 and u_2. An SBP-SAT discretization is

$$
\begin{aligned}
c_1 \mathbf{v}_{1,t} &= i D_1 \mathbf{v}_2 + \tau_{1\ell} H^{-1} \mathbf{e}_\ell (\mathbf{e}_\ell^T \mathbf{v}_1 - 0) + \tau_{1r} H^{-1} \mathbf{e}_r (\mathbf{e}_r^T \mathbf{v}_1 - 0) \\
c_2 \mathbf{v}_{2,t} &= -i D_1 \mathbf{v}_2 + \tau_{2\ell} H^{-1} \mathbf{e}_\ell (\mathbf{e}_\ell^T \mathbf{v}_1 - 0) + \tau_{2r} H^{-1} \mathbf{e}_r (\mathbf{e}_r^T \mathbf{v}_1 - 0)
\end{aligned}
$$

where the four scalar parameters $\tau_{1\ell}$, τ_{1r}, $\tau_{2\ell}$, and τ_{2r} would need to be selected such that the discretization is stable.

Alternative 2:

Equivalently, we may write the discretization on matrix-vector form by extending all operators to a system of equations. Let I_n denote the $n \times n$ identity matrix. Let $\mathbf{v}$ be the discrete solution vector:

$$
\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in \mathbb{C}^{2m+2}.
$$

We extend the SBP-related operators to a 2-component system:

$$
\bar{D}_1 = I_2 \otimes D_1, \quad \bar{H} = I_2 \otimes H, \quad \bar{e}_{\ell,r} = I_2 \otimes e_{\ell,r}
$$

and extend the coefficient matrices to $m+1$ grid points:

$$
\bar{C} = C \otimes I_{m+1}, \quad \bar{B} = B \otimes I_{m+1}.
$$

The SBP-SAT discretization can be written

$$
\bar{C} \mathbf{v}_t = \bar{B} \bar{D}_1 \mathbf{v} + \bar{H}^{-1} \begin{bmatrix} \tau_{1\ell} \mathbf{e}_\ell \\ \tau_{2\ell} \mathbf{e}_\ell \end{bmatrix} (\mathbf{e}_\ell^T \mathbf{v}_1 - 0) + \bar{H}^{-1} \begin{bmatrix} \tau_{1r} \mathbf{e}_r \\ \tau_{2r} \mathbf{e}_r \end{bmatrix} (\mathbf{e}_r^T \mathbf{v}_1 - 0)
$$

2. The second-order wave equation in a bounded 2D domain $\Omega \subset \mathbb{R}^2$ is given by

$$
\begin{aligned}
&\phi_{tt} = c^2 \nabla \cdot \nabla \phi, && \vec{x} \in \Omega, && t > 0, \\
&\frac{\partial \phi}{\partial \hat{\mathbf{n}}} + \alpha \phi_t = 0, && \vec{x} \in \partial\Omega, && t > 0, \\
&\phi = \phi_0, && \vec{x} \in \Omega, && t = 0, \\
&\phi_t = \varphi_0, && \vec{x} \in \Omega, && t = 0,
\end{aligned}
\tag{2}
$$

where $c = c(\vec{x}) > 0$, $\alpha \in \mathbb{R}$, $\partial\Omega$ denotes the boundary of Ω, and $\hat{\mathbf{n}}$ denotes the outward unit normal. We assume that the solution is real.

(a) Consider the case $\alpha = 0$. Prove that the IBVP (2) conserves energy. (2p)

(b) For $\alpha > 0$, prove that the IBVP (2) dissipates energy. (1p)

(a) We divide by c^2 and then apply the energy method, which amounts to multiplying the PDE by ϕ_t and integrating.

$$\left(\phi_t, c^{-2}\phi_{tt}\right) = \left(\phi_t, \nabla \cdot \nabla\phi\right) = \oint_{\partial\Omega} \phi_t \frac{\partial \phi}{\partial \hat{\mathbf{n}}} \, \mathrm{d}S - \left(\nabla\phi_t, \nabla\phi\right),$$

where we used the integration-by-parts formula for multiple dimensions (see attached collection of formulae). Since the inner product is symmetric:

$$\left(\phi_t, c^{-2}\phi_{tt}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_t, c^{-2}\phi_t\right), \quad \left(\nabla\phi_t, \nabla\phi\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla\phi, \nabla\phi\right).$$

We have derived

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\phi_t, c^{-2}\phi_t\right) + \frac{1}{2}\left(\nabla\phi, \nabla\phi\right)\right) = \oint_{\partial\Omega} \phi_t \frac{\partial \phi}{\partial \hat{\mathbf{n}}} \, \mathrm{d}S.$$

Recognizing the left-hand side as the rate of change of energy, we define

$$E = \frac{1}{2}\left(\phi_t, c^{-2}\phi_t\right) + \frac{1}{2}\left(\nabla\phi, \nabla\phi\right).$$

We have derived the energy rate

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \oint_{\partial\Omega} \phi_t \frac{\partial \phi}{\partial \hat{\mathbf{n}}} \, \mathrm{d}S.$$

With $\alpha = 0$, the boundary condition reads

$$\frac{\partial \phi}{\partial \hat{\mathbf{n}}} = 0,$$

which yields

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0,$$

which shows that energy is conserved.

(b) With $\alpha > 0$, the boundary condition reads

$$\frac{\partial \phi}{\partial \hat{\mathbf{n}}} = -\alpha\phi_t,$$

which yields the energy rate

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\oint_{\partial\Omega} \alpha\phi_t^2 \, \mathrm{d}S \leq 0,$$

which shows that energy is dissipated.

3. A 1D version of the second-order wave equation with constant wave speed is

$$\begin{aligned}
&\phi_{tt} = c^2\phi_{xx}, && x \in (0, L) && t > 0,\\
&\alpha\phi_t - \phi_x = 0, && x = 0, && t > 0,\\
&\alpha\phi_t + \phi_x = 0, && x = L, && t > 0,\\
&\phi = \phi_0, && x \in [0, L], && t = 0,\\
&\phi_t = \varphi_0, && x \in [0, L], && t = 0,
\end{aligned} \tag{3}$$

where $c > 0$ is constant, $L > 0$, and $\alpha \in \mathbb{R}$. We assume that the solution is real.

(a) State an SBP-SAT discretization of the PDE and boundary conditions. You do not need to prove stability. You may use one unspecified scalar penalty parameter per boundary. You may solve the problem with $\alpha = 0$ for 1 point out of 2. (2p)

(b) Select appropriate penalty parameters and prove that your SBP-SAT scheme is stable for $\alpha \geq 0$. You may solve the problem with $\alpha = 0$ for 2 points out of 3. (3p)

(a) Define a grid of $m + 1$ points with grid spacing $h = \frac{L}{m}$:

$$x_j = jh, \quad j = 0, 1 \ldots, m.$$

Let D_2 be a corresponding SBP operator for the first derivative (see attached formulae sheet) and let $\boldsymbol{\phi}$ be the discrete solution vector. An SBP-SAT discretization reads

$$c^{-2}\boldsymbol{\phi}_{tt} = D_2\boldsymbol{\phi} + \tau_\ell H^{-1}\mathbf{e}_\ell(\mathbf{d}_\ell^T\boldsymbol{\phi} - \alpha\mathbf{e}_\ell^T\boldsymbol{\phi}_t) + \tau_r H^{-1}\mathbf{e}_r(\mathbf{d}_r^T\boldsymbol{\phi} + \alpha\mathbf{e}_r^T\boldsymbol{\phi}_t),$$

where τ_ℓ and τ_r are scalar penalty parameters that will need to be selected such that the discretization is stable.

(b) The discrete energy method: multiply by $\boldsymbol{\phi}_t^T H$.

$$\left(\boldsymbol{\phi}_t, c^{-2}\boldsymbol{\phi}_{tt}\right)_H = (\boldsymbol{\phi}_t, D_2\boldsymbol{\phi})_H + \tau_\ell(\mathbf{e}_\ell^T\boldsymbol{\phi}_t)\left(\mathbf{d}_\ell^T\boldsymbol{\phi} - \alpha\mathbf{e}_\ell^T\boldsymbol{\phi}_t\right) + \tau_r(\mathbf{e}_r^T\boldsymbol{\phi}_t)\left(\mathbf{d}_r^T\boldsymbol{\phi} + \alpha\mathbf{e}_r^T\boldsymbol{\phi}_t\right)$$

By the SBP properties of D_2, we have

$$(\boldsymbol{\phi}_t, D_2\boldsymbol{\phi})_H = (\mathbf{e}_r^T\boldsymbol{\phi}_t)(\mathbf{d}_r^T\boldsymbol{\phi}) - (\mathbf{e}_\ell^T\boldsymbol{\phi}_t)(\mathbf{d}_\ell^T\boldsymbol{\phi}) - \boldsymbol{\phi}_t^T M\boldsymbol{\phi}.$$

Gather terms to obtain

$$\begin{aligned}
\left(\boldsymbol{\phi}_t, c^{-2}\boldsymbol{\phi}_{tt}\right)_H = &-\boldsymbol{\phi}_t^T M\boldsymbol{\phi} + (\mathbf{e}_\ell^T\boldsymbol{\phi}_t)(\mathbf{d}_\ell^T\boldsymbol{\phi})(-1 + \tau_\ell) + (\mathbf{e}_r^T\boldsymbol{\phi}_t)(\mathbf{d}_r^T\boldsymbol{\phi})(1 + \tau_r)\\
&- \alpha\tau_\ell(\mathbf{e}_\ell^T\boldsymbol{\phi})^2 + \alpha\tau_r(\mathbf{e}_r^T\boldsymbol{\phi})^2.
\end{aligned}$$

We can make the indefinite boundary terms vanish by setting

$$\tau_\ell = 1, \quad \tau_r = -1.$$

Noting that

$$\left(\boldsymbol{\phi}_t, c^{-2}\boldsymbol{\phi}_{tt}\right)_H = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\phi}_t, c^{-2}\boldsymbol{\phi}_t\right)_H,$$

$$\boldsymbol{\phi}_t^T M\boldsymbol{\phi} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^T M\boldsymbol{\phi},$$

we then have the energy rate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\boldsymbol{\phi}_t, c^{-2}\boldsymbol{\phi}_t\right)_H + \frac{1}{2}\boldsymbol{\phi}^T M\boldsymbol{\phi}\right) = -\alpha(\mathbf{e}_\ell^T\boldsymbol{\phi})^2 - \alpha(\mathbf{e}_r^T\boldsymbol{\phi})^2,$$

where

$$\frac{1}{2}\left(\boldsymbol{\phi}_t, c^{-2}\boldsymbol{\phi}_t\right)_H + \frac{1}{2}\boldsymbol{\phi}^T M\boldsymbol{\phi} =: E_h$$

is a discrete energy. The discrete energy rate mimics the energy rate for the continuum problem (see the previous problem). Energy is dissipated for $\alpha > 0$ and conserved for $\alpha = 0$. This shows that the SBP-SAT scheme with $\tau_\ell = 1$, $\tau_r = -1$ is stable for $\alpha \geq 0$.

4. Consider the heat equation with constant coefficients in the interval $\mathcal{I} = (0, L)$:

$$\begin{aligned} u_t = au_{xx}, &\quad x \in \mathcal{I}, \quad t > 0, \\ u = g, &\quad x = 0, \quad t > 0, \\ u_x = 0, &\quad x = L, \quad t > 0, \\ u = f, &\quad x \in \mathcal{I}, \quad t = 0, \end{aligned} \tag{4}$$

where g, $a > 0$ and $L > 0$ are real constants. We assume that the solution is real.

(a) Derive the weak form of (4) with appropriate spaces. (2p)

(b) State the finite element approximation of the weak form, using appropriate spaces of piecewise linear functions on a uniform mesh of n intervals. (2p)

(c) Derive the system of ODE corresponding to the finite element approximation. You do not need to evaluate any nonzero integrals, but you should indicate the structure of all matrices and vectors and show where they have nonzero entries. (3p)

(a) Let V be the space of functions $v(x,t)$ such that v and v_x are square-integrable over I for any t:

$$V = \left\{v(x,t) : \|v(\cdot,t)\| + \|v_x(\cdot,t)\| < \infty\right\}.$$

Further define two spaces of functions that additionally satisfy Dirichlet boundary conditions at $x = 0$:

$$V_0 = \{v \in V : v(0,t) = 0\}$$

$$V_g = \{v \in V : v(0,t) = g\}.$$

We seek a solution $u \in V_g$. To derive the weak form, we multiply the PDE by a test function $v \in V_0$ and integrate. We obtain

$$(v, u_t) = a\,(v, u_{xx}) = avu_x|_0^L - a\,(v_x, u_x)\,.$$

Using the boundary conditions, we obtain

$$(v, u_t) = -a\,(v_x, u_x)\,.$$

The weak form reads: Find $u \in V_g$ such that

$$(v, u_t) = -a\,(v_x, u_x) \quad \forall v \in V_0.$$

(b) Let V_h denote the set of functions $v(x,t)$ that are piecewise linear functions of x on a uniform mesh of n intervals. Further define two spaces of functions that additionally satisfy Dirichlet boundary conditions at $x = 0$:

$$V_{h,0} = \{v \in V_h : v(0,t) = 0\}$$

$$V_{h,g} = \{v \in V_h : v(0,t) = g\}.$$

The finite element approximation reads: Find $u_h \in V_{h,g}$ such that

$$(v, u_{h,t}) = -a\,(v_x, u_{h,x}) \quad \forall v \in V_{h,0}.$$

(c) Any $v \in V_{h,0}$ can be represented as a linear combination of the hat functions φ_i:

$$v(x,t) = \sum_{i=1}^{n} \alpha_i(t)\varphi_i(x).$$

We may satisfy the weak form for one hat function at a time,

$$(\varphi_i, u_{h,t}) = -a\,(\varphi_i', u_{h,x})\,, \quad i = 1, \ldots, n.$$

Any $u_h \in V_{h,g}$ satisfies

$$u_h(x,t) = g\varphi_0(x) + \sum_{j=1}^{n} \xi_j(t)\varphi_j(x).$$

for some unknown, time-dependent coefficients ξ_j. The derivatives of u_h are

$$u_{h,x} = g\varphi_0' + \sum_{j=1}^{n} \xi_j \varphi_j'$$

$$u_{h,t} = \sum_{j=1}^{n} \xi_j' \varphi_j$$

Inserting this ansatz leads to

$$\sum_{j=1}^{n} (\varphi_i, \varphi_j)\, \xi_j' = -ag\,(\varphi_i', \varphi_0') - \sum_{j=1}^{n} a\left(\varphi_i', \varphi_j'\right) \xi_j, \quad i = 1, \dots, n.$$

This is an $n \times n$ system of ODE:

$$M\xi' = A\xi + r$$

where M (mass matrix) and A (stiffness matrix) are tri-diagonal matrices, with elements

$$M_{ij} = (\varphi_i, \varphi_j)\,, \quad A_{ij} = -a\left(\varphi_i', \varphi_j'\right), \quad i = 1, \dots, n,\ j = 1, \dots, n,$$

and r is an $n \times 1$ vector, with elements

$$r_i = -ag\,(\varphi_i', \varphi_0')\,, \quad i = 1, \dots, n.$$

Since (φ_i', φ_0') equals 0 for $i > 1$, r only has one nonzero element, at the top.

5. Consider the linear system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & -32 & 4 \\ 1 & 0 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}.$$

(a) Perform one iteration using the Gauss-Seidel method. Use the starting guess $x_0 = [1, 1, 8]^T$. (1p)

(b) Do you expect the Gauss-Seidel method to converge for this linear system? Motivate your answer. (2p)

(a) The Gauss-Seidel iteration is

$$A_1 x_{k+1} = -A_2 x_k + b,$$

where

$$A_1 = L + D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -32 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$$A_2 = U = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Setting $k = 0$, inserting all values and solving for x_1 yields

$$x_1 = \begin{bmatrix} -12 \\ 1 \\ 4 \end{bmatrix}.$$

(b) The Gauss-Seidel method can be written as:

$$\mathbf{x}^{k+1} = R\mathbf{x}^k + \mathbf{c}, \quad k = 0, 1, \dots,$$

where

$$R = -A_1^{-1}A_2 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

The 1-norm and ∞-norm of R are

$$\|R\|_1 = \max_j \sum_{i=1}^{3} |R_{ij}| = 2.6250.$$

$$\|R\|_\infty = \max_i \sum_{j=1}^{3} |R_{ij}| = 2.$$

Both norms are greater than 1, so the iteration might not converge. To know for sure, we need to compute the spectral radius of R. We find that the eigenvalues are

$$\lambda_1 = \lambda_2 = 0, \ \lambda_3 = 0.5,$$

so the spectral radius (largest magnitude eigenvalue) is

$$\rho(R) = 0.5,$$

which shows that the iteration will actually converge!

Collection of formulae

Summation-by-parts operators

Consider a uniform grid of $m+1$ points, with grid spacing $h = \frac{L}{m}$. Let $\mathbf{e}_\ell$ and $\mathbf{e}_r$ denote the following vectors in $\mathbb{R}^{m+1}$:

$$\mathbf{e}_\ell = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_r = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Definition of D_1

A difference operator D_1 approximating $\partial/\partial x$ is a first-derivative SBP operator with quadrature matrix H if $H = H^T > 0$ and

$$HD_1 = \mathbf{e}_r\mathbf{e}_r^T - \mathbf{e}_\ell\mathbf{e}_\ell^T - D_1^T H.$$

Definition of D_2

A difference operator D_2 approximating $\partial^2/\partial x^2$ is a second-derivative SBP operator with quadrature matrix H if $H = H^T > 0$ and

$$HD_2 = \mathbf{e}_r\mathbf{d}_r^T - \mathbf{e}_\ell\mathbf{d}_\ell^T - M,$$

where $M = M^T \geq 0$, and $\mathbf{d}_\ell^T v \simeq u_x|_{x=0}$, $\mathbf{d}_r^T v \simeq u_x|_{x=L}$ are finite difference approximations of the first derivatives at the left and right boundary points.

Discrete inner product

Let $(\cdot,\cdot)_H$ denote the discrete inner product, defined by

$$(\mathbf{u},\mathbf{v})_H = \mathbf{u}^* H \mathbf{v}.$$

In the discrete inner product, the SBP operators satisfy

$$(\mathbf{u}, D_1\mathbf{v})_H = (\mathbf{e}_r^T\mathbf{u})^*(\mathbf{e}_r^T\mathbf{v}) - (\mathbf{e}_\ell^T\mathbf{u})^*(\mathbf{e}_\ell^T\mathbf{v}) - (D_1\mathbf{u},\mathbf{v})_H$$

and

$$(\mathbf{u}, D_2\mathbf{v})_H = (\mathbf{e}_r^T\mathbf{u})^*(\mathbf{d}_r^T\mathbf{v}) - (\mathbf{e}_\ell^T\mathbf{u})^*(\mathbf{d}_\ell^T\mathbf{v}) - \mathbf{u}^* M\mathbf{v}.$$

Integration by parts in multiple dimensions

Let Ω denote a bounded domain in $\mathbb{R}^n$, with boundary $\partial\Omega$ and outward unit normal $\hat{\mathbf{n}}$. For (sufficiently smooth) scalar functions $u, v, \alpha \in L^2(\Omega)$,

$$\int_{\Omega} u \nabla \cdot \alpha \nabla v \, \mathrm{d}\Omega = \oint_{\partial\Omega} u \alpha \frac{\partial v}{\partial \hat{\mathbf{n}}} \, \mathrm{d}S - \int_{\Omega} \nabla u \cdot \alpha \nabla v \, \mathrm{d}\Omega.$$

This relation follows from the divergence theorem.

Finite element methods

Given a mesh of n intervals

$$x_0 < x_1 < \ldots < x_n,$$

the usual basis functions (the hat functions) in the corresponding space of piecewise linear functions satisfy, for $i = 1, \ldots, n-1$,

$$(\varphi_i', \varphi_j') = \begin{cases} \dfrac{1}{h_i} + \dfrac{1}{h_{i+1}}, & i = j \\ -\dfrac{1}{h_{i+1}}, & j = i+1 \\ -\dfrac{1}{h_i}, & j = i-1 \\ 0, & |i-j| > 1 \end{cases}$$

where $h_i = x_i - x_{i-1}$. Further, for $i = 1, \ldots, n-1$:

$$(\varphi_i, \varphi_j) = \begin{cases} \dfrac{h_i}{3} + \dfrac{h_{i+1}}{3}, & i = j \\ \dfrac{h_{i+1}}{6}, & j = i+1 \\ \dfrac{h_i}{6}, & j = i-1 \\ 0, & |i-j| > 1 \end{cases}$$