1. Consider the following problem,

$$\begin{array}{ll} i\,\mathbf{u}_t = \mathbf{A}\mathbf{u}_x + \mathbf{V}\mathbf{u} & -1 \le x \le 1,\ t \ge 0, \\ L_l\mathbf{u} = 0, & x = -1,\ t \ge 0, \\ L_r\mathbf{u} = 0, & x = 1,\ t \ge 0, \\ \mathbf{u} = \mathbf{f}(x), & -1 \le x \le 1,\ t = 0, \end{array} \qquad (1)$$

where L_l and L_r are the boundary operators, $\underline{\mathbf{V} > 0}$, $\mathbf{f} = \mathbf{f}(x)$ is the initial data and

$$\mathbf{u} = \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix},\ \mathbf{A} = \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix}, \qquad V = V^* > 0\ ?$$

Let us assume $V = V^*$.

(a) What are the requirements for (1) to be well-posed, disregarding the boundary conditions, i.e. here assume $BT \le 0$. (1p)

The PDE is first order in time.

$\Longrightarrow$ One IC is correct.

$$iu_t = Au_x + Vu \iff u_t = \underbrace{-iA}_{B}u_x \underbrace{-iV}_{C}u$$

We have seen $u_t = Bu_x + Cu$ before.

This PDE is hyperbolic, and thus well posed, if B is diagonalizable with real eigenvalues ($\iff$ A diagonalizable with purely imaginary eigenvalues)

As long as C is bounded, $\|C\| < \infty$, the term Cu does not affect well-posedness, because it can at most cause exponential growth.

$B = B^* \Rightarrow B$ has only real eigenvalues.

$A = -A^* \Rightarrow A$ has purely imaginary eigenvalues.

$A = -A^*$ iff $Re(\alpha) = 0$. This guarantees well-posedness.

(b) Let $\alpha = 0$, and derive a set of well-posed boundary conditions for (1), thats leads to damping of energy. This means finding L_l and L_r. (1p)

Study $u_t = Bu_x + Cu$, where

$$B = -iA = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad B = B^*.$$

$$C = -iV, \quad C^* = -C \quad (\text{since } V^* = V)$$

Energy method

1. Mult. by u^* and integrate

$$(u, u_t) = (u, Bu_x) + (u, Cu) \quad (1)$$

2. IBP in (1)

$$(u, u_t) = u^* B u \Big|_{-1}^{1} - (u_x, Bu) + (u, Cu) \quad (2)$$

3. Conjugate of (1)

$$(u_t, u) = (Bu_x, u) + (Cu, u) = (u_x, B^*u) + (u, C^*u) \quad (3)$$

4. Add (2) and (3)

$$\frac{d}{dt}\|u\|^2 = (u_x, \underbrace{(B^* - B)}_{=0}u) + (u, \underbrace{(C + C^*)}_{=0}u) + \underbrace{u^* B u \Big|_{-1}^{1}}_{BT}$$

$$BT = u^* B u \Big|_{-1}^{1} = \left[-i u^{(1)*} u^{(2)} + i u^{(2)*} u^{(1)} \right]_{-1}^{1}$$

The eigenvalues of B are ± 1 $\Rightarrow$ we need one BC at each boundary.

Ansatz: $$u^{(1)} = \beta_L u^{(2)}, \quad x = -1$$
$$u^{(1)} = \beta_R u^{(2)}, \quad x = 1$$

Try to find $\beta_{L,R}$ that yield dissipation.

$$BT = +i\beta_L^* u^{(2)*} u^{(2)} - i u^{(2)*} \beta_L u^{(2)} \Big|_{-1}$$
$$- i\beta_R^* u^{(2)*} u^{(2)} + i u^{(2)*} \beta_R u^{(2)} \Big|_{1}$$

$$= -i(\beta_L - \beta_L^*) |u^{(2)}|^2 \Big|_{-1} + i(\beta_R - \beta_R^*) |u^{(2)}|^2 \Big|_{1}$$

$$i(\beta - \beta^*) = i\, 2i\, \mathrm{Im}(\beta) = -2\,\mathrm{Im}(\beta)$$

need $\mathrm{Im}(\beta_L) < 0$, $\mathrm{Im}(\beta_R) > 0$

Example: $$u^{(1)} = -i u^{(2)}, \quad x = -1$$
$$u^{(1)} = i u^{(2)}, \quad x = 1$$

(c) Let $\alpha = 0$, and derive two sets of well-posed boundary conditions for (1), thats leads to conservation of energy. This means finding L_l and L_r. (1p)

$u^{(1)} = 0$ yields energy conservation

$u^{(2)} = 0$ ——— " ———

4 combinations:

$$\begin{cases} u^{(1)}(-1,t) = 0 \\ u^{(1)}(1,t) = 0 \end{cases} \qquad \begin{cases} u^{(1)}(-1,t) = 0 \\ u^{(2)}(1,t) = 0 \end{cases}$$

$$\begin{cases} u^{(2)}(-1,t) = 0 \\ u^{(1)}(1,t) = 0 \end{cases} \qquad \begin{cases} u^{(2)}(-1,t) = 0 \\ u^{(2)}(1,t) = 0 \end{cases}$$

(d) Derive an SBP-~~Projection~~ SAT approximation of (1), with any set of the well-posed boundary conditions derived in (c), where $\alpha = 0$. (1p)

Choose $\begin{cases} u^{(1)}(-1,t) = 0 \\ u^{(1)}(1,t) = 0 \end{cases}$.

Grid of $m+1$ points: $x_j = -1 + jh, \quad j = 0, 1, \dots, m$

$$h = \frac{1-(-1)}{m} = \frac{2}{m}$$

SBP operator: $D_1 \leftarrow (m+1) \times (m+1)$

$$HD_1 = e_r e_r^T - e_l e_l^T - D_1^T H$$

<u>Extend to system.</u>

Solution vector: $v = \left[v_0^{(1)}, v_1^{(1)}, \dots, v_m^{(1)}, v_0^{(2)}, \dots, v_m^{(2)}\right]^T$

$\bar{D}_1 = I_2 \otimes D_1, \quad \bar{H} = I_2 \otimes H, \quad \bar{e}_{l,r} = I_2 \otimes e_{l,r}$

$\bar{A} = A \otimes I_{m+1}$, $\bar{V} = V \otimes I_{m+1}$ (assuming V constant.)

$$i v_t = \bar{A} \bar{D}_1 v + \bar{V} v + \bar{H}^{-1} \begin{bmatrix} \tau_{\ell 1} e_\ell \\ \tau_{\ell 2} e_\ell \end{bmatrix} (v_0^{(1)} - 0) \underbrace{+ \bar{H}^{-1} \begin{bmatrix} \tau_{r1} e_r \\ \tau_{r2} e_r \end{bmatrix} (v_m^{(1)} - 0)}_{SAT}$$

(e) Show stability for the SBP-~~Projection~~ SAT approximation in (d). (2p)

Rewrite: $v_t = \bar{B} \bar{D}_1 v + \bar{C} v - i\,SAT$

where $\bar{B} = -i\bar{A}$, $\bar{C} = -i\bar{V}$

Energy method

1. Multiply by $v^* \bar{H}$

$$(v, v_t)_{\bar{H}} = (v, \bar{B} D_1 v)_{\bar{H}} + (v, \bar{C} v) \overbrace{- v^* \bar{H} i\,SAT}^{X} \quad (1)$$

2. SBP in (1)

$$(v, v_t)_{\bar{H}} = \overbrace{(\bar{e}_r^T v)^* B (\bar{e}_r^T v) - (\bar{e}_\ell^T v)^* B (e_\ell^T v)}^{BT} - (\bar{D}_1 v, \bar{B} v)_{\bar{H}}$$
$$+ (v, \bar{C} v) + X \quad (2)$$

3. Conjugate of (1)

$$(v_t, v)_{\bar{H}} = (\bar{B}\bar{D}_1 v, v)_{\bar{H}} + (\bar{C} v, v)_{\bar{H}} + X^*$$

$$= (\bar{D}_1 v, \bar{B}^* v)_{\bar{H}} + (v, \bar{C}^* v)_{\bar{H}} + X^*$$

4. Add (2) and (3)

$$\frac{d}{dt}\|v\|^2_{\bar{H}} = \Big(\bar{D}_1 v, \big(\underbrace{\bar{B}^* - \bar{B}}_{0}\big) v\Big)_{\bar{H}} + \Big(v, \big(\underbrace{\bar{C} + \bar{C}^*}_{0}\big) v\Big)_{\bar{H}}$$

$$+ BT + X + X^*$$

$$BT = (\bar{e}_r^T v)^* B (\bar{e}_r^T v) - (\bar{e}_\ell^T v)^* B (\bar{e}_\ell^T v)$$

$$= -i\, v_m^{(1)*} v_m^{(2)} + i\, v_m^{(2)*} v_m^{(1)} + i\, v_0^{(1)*} v_0^{(2)} - i\, v_0^{(2)*} v_0^{(1)}$$

$$X = -v^* \bar{H}\, i\, SAT = -i\left(v_0^{(1)*} \tau_{\ell 1} v_0^{(1)} + v_0^{(2)*} \tau_{\ell 2} v_0^{(1)}\right)$$

$$- i\left(v_m^{(1)*} \tau_{r1} v_m^{(1)} + v_m^{(2)*} \tau_{r2} v_m^{(1)}\right)$$

$$= -i\tau_{\ell 1} |v_0^{(1)}|^2 - i\tau_{\ell 2} v_0^{(2)*} v_0^{(1)}$$

$$- i\tau_{r1} |v_m^{(1)}|^2 - i\tau_{r2} v_m^{(2)*} v_m^{(1)}$$

$$X^* = + i\tau_{\ell 1}^* |v_0^{(1)}|^2 + i\tau_{\ell 2}^* v_0^{(2)} v_0^{(1)*}$$

$$+ i\tau_{r1}^* |v_m^{(1)}|^2 + i\tau_{r2}^* v_m^{(2)} v_m^{(1)*}$$

$$BT + X + X^* = V_m^{(1)*} V_m^{(2)} \left(-i + i\tau_{r2}^*\right) \longrightarrow \tau_{r2} = 1$$

$$+ V_m^{(2)*} V_m^{(1)} \left(i - i\tau_{r2}\right) \longrightarrow \tau_{r2} = 1$$

$$+ V_0^{(1)*} V_0^{(2)} \left(i + i\tau_{\ell 2}^*\right) \longrightarrow \tau_{\ell 2} = -1$$

$$+ V_0^{(2)*} V_0^{(1)} \left(-i - i\tau_{\ell 2}\right) \longrightarrow \tau_{\ell 2} = -1$$

$$+ \left|V_0^{(1)}\right|^2 i\left(\tau_{\ell 1}^* - \tau_{\ell 1}\right) \longrightarrow i\left(\tau_{\ell 1}^* - \tau_{\ell 1}\right) \leq 0$$

$$+ \left|V_m^{(1)}\right|^2 i\left(\tau_{r1}^* - \tau_{r1}\right) \longrightarrow i\left(\tau_{r1}^* - \tau_{r1}\right) \leq 0$$

$$i(\tau^* - \tau) = -i2i\,\mathrm{Im}(\tau) = 2\,\mathrm{Im}(\tau)$$

Need $\tau_{r2} = 1$, $\tau_{\ell 2} = -1$, $\mathrm{Im}(\tau_{\ell 1}) \leq 0$, $\mathrm{Im}(\tau_{\ell 2}) \leq 0$

With these parameters, the scheme is stable because we obtain

$$\frac{d}{dt}\|v\|_H^2 \leq 0.$$

(f) Explain why Euler forward (RK1) is not a suitable time-integrator for the SBP-Projection approximation derived in (e). Propose a more suitable time-integrator and give a rough estimate how to chose the time-step to obtain stability for an arbitrary grid-spacing h. (2p)

With $V=0$, the PDE is $u_t = Bu_x$,

B has real eigenvalues.

The Fourier coefficients satisfy the ODE: $\frac{d\hat{u}_k}{dt} = \underbrace{Bik}_{\text{purely imaginary EV.}}\hat{u}_k$

The semi-discrete ODE will have approximately the same behavior, $v_t = Mv$, where M has eigenvalues close to the imaginary axis. The stability region of RK1 does not cover the imaginary axis!

We can use RK4 instead!

The PDE is hyperbolic

$\Longrightarrow$ standard CFL condition $\Delta t \lesssim \underbrace{C}_{\text{dimensionless constant.}} \frac{h}{c}$

where $C \approx 1$ and c is the largest wave speed. Here the EV of B are ± 1, so $c = 1$.

3. Consider the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathbf{A} = \begin{pmatrix} y & 0 & 1 \\ 0 & -8 & 1 \\ z & 0 & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

where y and z are some parameters.

(a) Write down the Gauss-Seidel method to solve the system and do one iteration using the starting vector $\boldsymbol{x}^0 = (1, 1, 1)^\top$. (1p)

(b) For which values of y and z the Gauss-Seidel method can be expected to converge for this problem? Motivate your answer.
Hint: note that the first element of the last row of the inverse of the lower triangular matrix would be $\cancel{-\frac{b}{a}}$. $-\frac{z}{y}$ (3p)

Gauss - Seidel:

$$A = L + D + U = A_1 + A_2$$

$$A_1 = L + D, \quad A_2 = U$$

$$A_1 x_{k+1} = -A_2 x_k + b$$

Can be written as

$$x_{k+1} = R x_k + c, \quad \text{where} \quad R = -A_1^{-1} A_2$$

$$c = A_1^{-1} b.$$

Here: $A_1 = \begin{pmatrix} y & & \\ & -8 & \\ z & & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} & & 1 \\ & & 1 \\ & & 0 \end{pmatrix}$

$$A_1^{-1} = \begin{pmatrix} 1/y & & \\ & -\frac{1}{8} & \\ -\frac{z}{y} & & 1 \end{pmatrix}, \quad R = \begin{pmatrix} & & -1/y \\ & & 1/8 \\ & & z/y \end{pmatrix}$$

$$c = \begin{pmatrix} 1/y \\ -1/4 \\ -z/y + 3 \end{pmatrix}$$

a) $x_1 = Rx_0 + c = \ldots = \begin{pmatrix} 0 \\ -\frac{1}{8} \\ 3 \end{pmatrix}$

b Guaranteed conv. if $\|R\|_p < 1$, for any p.

$$\|R\|_\infty = \{\text{max row sum}\}$$

$$= \max\left(\frac{1}{|y|}, \frac{1}{8}, \frac{|z|}{|y|}\right)$$

Conv. if $|y| > 1$ and $|z| < |y|$.

Conv. $\Longleftrightarrow$ $\rho(R) < 1$.

The eigenvalues of R are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \frac{z}{y}$

$\Rightarrow$ Convergence for any x_0 iff $|z| < |y|$.

(c) For which values of y and z can one use the Conjugate-Gradient method to solve the above problem? Motivate your answer. (1p)

CG requires $A = A^* > 0$.

No choice of y, z yields $A = A^*$.

$\Rightarrow$ CG is not applicable.

2. Consider the following problem in $\Omega = (0,1)$:

$$\begin{aligned} u_t - u_{xx} + \alpha u_x = 0, &\quad x \in \Omega,\ t > 0, \\ u(0,t) = 1, &\quad t > 0, \\ u'(1,t) = 1, &\quad t > 0. \\ u(x,t) = u_0(x), &\quad x \in \Omega, \end{aligned} \tag{2}$$

where $\alpha > 0$, and $u_0(x)$ is a given initial condition.

(a) Write down a weak and finite element formulations for (2) with appropriate spaces. (2p)

$$V = \{ v(x,t) : \|v(\cdot,t)\| + \|v_x(\cdot,t)\| < \infty \}$$

$$V_0 = \{ v \in V : v(0,t) = 0 \}$$

$$V_1 = \{ v \in V : v(0,t) = 1 \}$$

Multiply by test function $v \in V_0$:

$$\begin{aligned} (v, u_t) + (v, \alpha u_x) &= (v, u_{xx}) \\ &= v u_x \Big|_0^1 - (v_x, u_x) \qquad \text{Use BC} \\ &= v(1,t) - (v_x, u_x) \end{aligned}$$

Weak form Find $u \in V_1$ such that

$$(v, u_t) + (v, \alpha u_x) = v(1,t) - (v_x, u_x) \quad \forall v \in V_0$$

Let V_h be a suitable finite element space, for example the space of piecewise linears: $V_h = \{ v \in C_0(\Omega) : v|_{I_i} \in P_1(I_i) \}$

Define $V_{h,0} = \{ v \in V_h : v(0,t) = 0 \}$

$V_{h,1} = \{ v \in V_h : v(0,t) = 1 \}$

FEM: Find $u_h \in V_{h,1}$ such that

$$(v, u_{h,t}) + (v, \alpha u_{h,x}) = v(1,t) - (v_x, u_{h,x}) \quad \forall v \in V_{h,0}$$

(b) Now split the interval into N equally spaced sub-intervals: $0 = x_0 < x_1 < \ldots < x_N = 1$. Construct the corresponding system of ordinary differential equations. Compute the elements of the resulting matrices.

Hint: you can use the following Simpson's rule:

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6}\Big(f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\Big),$$

to compute the elements of the mass matrix $\int_0^1 \varphi_j \varphi_i \,\mathrm{d}x$. (4p)

Since $v \in V_{h,0}$: $v(x,t) = \sum_{i=1}^{N} \beta_i(t) \varphi_i(x)$

where φ_i are the hat functions.

$\Rightarrow (\varphi_i, u_{h,t}) + (\varphi_i, \alpha u_{h,x}) = \varphi_i(1) - (\varphi_i', u_{h,x})$, $i = 1, \dots, N$

Since $u_h \in V_{h,1}$: $u_h(x,t) = \varphi_0(x) + \sum_{j=1}^{N} \xi_j(t) \varphi_j(x)$

$$u_{h,t} = \sum_{j=1}^{N} \xi_j' \varphi_j$$

$$\Rightarrow \sum_{j=1}^{N} \overbrace{(\varphi_i, \varphi_j)}^{M_{ij}} \xi_j' + \overbrace{(\varphi_i, \alpha \varphi_0')}^{z_i} + \sum_{j=1}^{N} \overbrace{(\varphi_i, \alpha \varphi_j')}^{B_{ij}} \xi_j$$

$$= \underbrace{\varphi_i(1)}_{r_i} - \underbrace{(\varphi_i', \varphi_0')}_{a_i} - \sum_{j=1}^{N} \underbrace{(\varphi_i', \varphi_j')}_{A_{ij}} \xi_j, \quad i = 1, \dots, N$$

$\Rightarrow$ System of ODE :

$$M \xi'(t) + (B + A) \xi(t) = b$$

where $b = r - a - z$.

If α depends on x, then all terms with α are typically evaluated using quadrature.

Simpson's method is exact for 2nd degree polynomials, so we might as well compute integrals in the mass matrix exactly.

To compute A_{ij}, see lecture notes.

M_{ij} and other integrals can be computed similarly.

(c) Discretize the ODE system in (b) using the explicit Euler method. (2p)

Let ξ^n correspond to time $t_n = n\Delta t$,

Fwd Euler: $M \dfrac{\xi^{n+1} - \xi^n}{\Delta t} + (B+A)\xi^n = b$