Solution

Q1: a) LU- factorization of A.

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 4 & 2 \\ 2 & 0 & 10 \end{pmatrix} \longrightarrow E_{31} A = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & -\frac{4}{5} & \frac{48}{5} \end{pmatrix}$$

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{5} & 0 & 1 \end{pmatrix} ; \quad E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{5} & 1 \end{pmatrix}$$

$$E_{32} E_{31} A = \underbrace{\begin{pmatrix} 5 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 10 \end{pmatrix}}_{} = U;$$

$$L = (E_{32} \cdot E_{31})^{-1} = E_{31}^{-1} \cdot E_{32}^{-1} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{5} & -\frac{1}{5} & 1 \end{pmatrix}}_{}.$$

b) Now solve the linear system:

1) given $\bar{b}$ solve $L\bar{y} = \bar{b}$

2) given $\bar{y}$ solve $U\bar{x} = \bar{y}$

1) $$\bar{y} = L^{-1} \bar{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} & 1 \end{pmatrix} \begin{pmatrix} -12 \\ 16 \\ 12 \end{pmatrix} = \begin{pmatrix} -12 \\ 16 \\ 20 \end{pmatrix}$$

2) $$\begin{pmatrix} 5 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -12 \\ 16 \\ 20 \end{pmatrix} \Rightarrow \quad x_1 = -4, \quad x_2 = 3, \quad x_3 = 2.$$

Q2. Analyze the Jacobi method applied to solve $A\bar{x}=\bar{b}$.

Since A is diagonally dominant the Jacobi iteration converges. We have:

$$A = D + L + R \Rightarrow A\bar{x}=\bar{b} \Rightarrow \bar{x} = \underbrace{-D^{-1}(L+R)}_{M}\bar{x} + \underbrace{D^{-1}\bar{b}}_{\bar{c}}$$

Now, start with $\bar{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and iterate:

$$\bar{x}_{k+1} = M\bar{x}_k + \bar{c} \quad \text{for } k=0,1,\ldots$$

$$M = -\begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix}\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & -\frac{1}{2} \\ -\frac{1}{5} & 0 & 0 \end{pmatrix},$$

$$\bar{c} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix}\begin{pmatrix} -12 \\ 16 \\ 12 \end{pmatrix} = \begin{pmatrix} -\frac{12}{5} \\ 4 \\ \frac{12}{10} \end{pmatrix}.$$

$$\bar{x}_1 = M\cdot\bar{0} + \bar{c} = \begin{pmatrix} -12/5 \\ 4 \\ 12/10 \end{pmatrix};$$

$$\bar{x}_2 = \begin{pmatrix} 0 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & -\frac{1}{2} \\ -\frac{1}{5} & 0 & 0 \end{pmatrix}\begin{pmatrix} -12/5 \\ 4 \\ 12/10 \end{pmatrix} + \begin{pmatrix} -\frac{12}{5} \\ 4 \\ 12/5 \end{pmatrix} = \begin{pmatrix} -212/50 \\ 34/10 \\ 84/50 \end{pmatrix}.$$

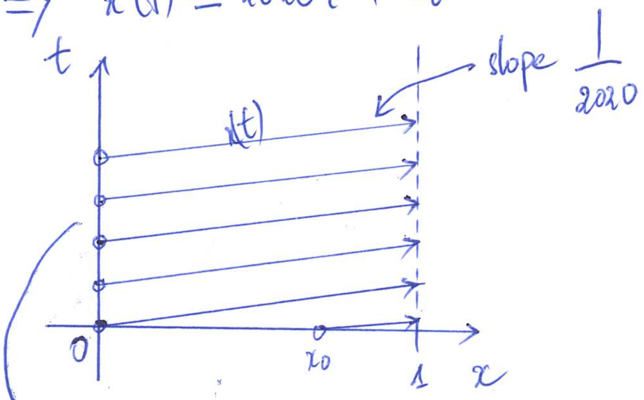
Question 3.

a) The method of Characteristics

$$\frac{du}{dt}(x(t),t) = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial t}\cdot\underbrace{\frac{dt}{dt}}_{=1}$$

Along the line $\underbrace{\frac{dx}{dt} = 2020}_{\text{from the PDE}}$, and $\frac{du}{dt} = 0$.

$\Rightarrow \quad x(t) = 2020t + x_0$



need to set BC at $x = 0$.

b) Local truncation error

$$(x) \quad \tau = \frac{1}{\Delta t}\Big(u(x_j, t_{n+1}) - u(x_j, t_{n+1} - \Delta t)\Big) + \frac{2020}{2\Delta x}\Big(u(x_j + \Delta x, t_{n+1}) - u(x_j - \Delta x, t_{n+1})\Big)$$

$$= \underbrace{u_t(x_j, t_{n+1}) + 2020\, u_x(x_j, t_{n+1})}_{PDE \Rightarrow\ =0} - \frac{\Delta t}{2} u_{tt}(x_j, t_{n+1}) + \frac{\Delta x^2}{3} u_{xxx}(x_j, t_{n+1}) + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^3)$$

$$= \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2) \longrightarrow 0 \text{ as } \Delta x, \Delta t \to 0$$

First order in time, second order in space

(3) $$\tau = \frac{1}{\Delta t}\Big(u(x_j, t_n+\Delta t) - u(x_j, t_n)\Big) + 2020 \cdot \frac{1}{2\Delta x}\Big[u(x_j - 2\Delta x, t_n) - 4u(x_j - \Delta x, t_n) + 3u(x_j, t_n)\Big]$$

$$= \underbrace{u_t(x_j, t_n) + 2020\, u_x(x_j, t_n)}_{0} + \frac{\Delta t}{2} u_{tt} - 2020 \frac{\Delta x^2}{3} u_{xxx} + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^3)$$

$$= \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2) \longrightarrow 0 \quad \text{as } \Delta x, \Delta t \to 0$$

first order in time, second order in space.

c) Insert ansatz $u_j^n = q^n e^{iwx_j}$ into (2)

$$q = 1 - \frac{2020\Delta t}{2\Delta x} q \underbrace{\left(e^{iw\Delta x} - e^{-iw\Delta x}\right)}_{2i\sin(w\Delta x)}$$

$$\Rightarrow q = \frac{1}{1 + \frac{2020\Delta t}{2\Delta x} i \sin(w\Delta x)}$$

$$\Rightarrow |q|^2 = \frac{1}{1 + \left(\frac{2020\Delta t}{2\Delta x}\right)^2 \sin^2(w\Delta x)} \leq 1 \quad \forall w, \Delta t, \Delta x$$

The method is <u>unconditionally stable</u>!

*) Lax-Richmyer equivalence theorem

consistency + stability $\Leftrightarrow$ convergence

d) Method (2)

+) Approximating u_x by a central difference scheme

⇒ dispersion error, unphysical oscillations especially in presence of shocks

+) Can take large time step because it's unconditionally stable

+) Implicit scheme ⇒ need to solve a system in each time step

Method (3)

+) Upwind scheme ⇒ less dispersion

+) One needs to make sure that step sizes in time is small enough.

+) Explicit scheme ⇒ less calculation in each time step

Q4.

(DE) $\begin{cases} \partial_x \frac{Uu}{2} - \partial_x(\varepsilon \partial_x u) = f, & 0<x<1 \\ u(0)=1, \\ u'(1)=0. \end{cases}$

(a) Find $u \in V_g = \{v : \|v\|^2<\infty, \|v'\|^2<\infty, v(0)=1\}$ such that

$$\frac{U}{2}\int_0^1 \partial_x u v \, dx + \int_0^1 \varepsilon \partial_x u \partial_x v \, dx - \varepsilon \partial_x u v \Big|_0^1 = \int_0^1 f v \, dx$$

$\forall v \in V_0 = \{v : \|v\|^2<\infty, \|v'\|^2<\infty, v(0)=0\}.$

Now apply the boundary condition:

$$-\varepsilon \partial_x u v \Big|_0^1 = -\varepsilon(\partial_x u(1) \overset{=0}{v(1)} - \partial_x \overset{=0}{u(0)} v(0)) = 0.$$

(WF) Find $u \in V_g$ such that

$$\frac{U}{2}\int_0^1 \partial_x u v \, dx + \int_0^1 \varepsilon \partial_x u \partial_x v \, dx = \int_0^1 f v \, dx \quad \forall v \in V_0$$

(b) Let us construct a triangulation of $[0,1]$:

$$0 = x_0 < x_1 < \dots < x_N = 1 .$$

Denote: $I_i = (x_{i-1}, x_i)$, $i = \overline{1,N}$. Then construct a subspace of V_g:

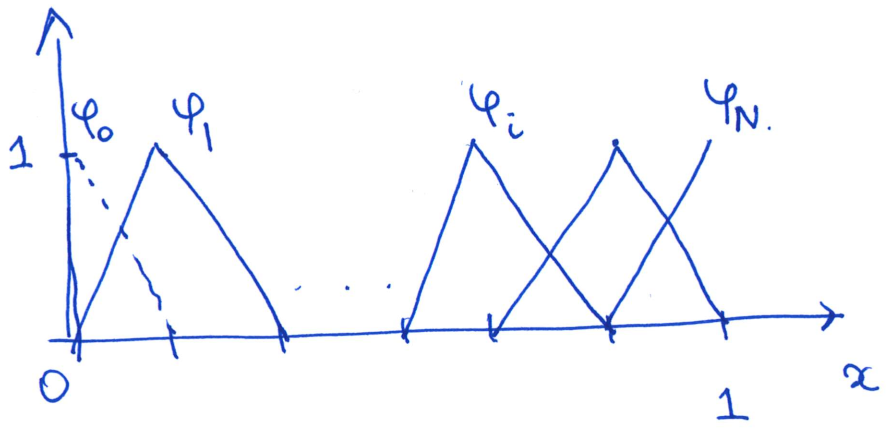
$$V_{g,h} = \{ v : v \in C^0(0,1),\ v|_{I_i} \in P^1(I_i),\ \forall i = \overline{1,N},\ v(0) = 1 \}$$

and similarly:

$$V_{0,h} = \{ v : v \in C^0(0,1),\ v|_{I_i} \in P^1(I_i), \forall i = \overline{1,N},\ v(0) = 0 \}.$$

$$\text{(GFEM)} \begin{cases} \text{Find } u_h \in V_{g,h} \text{ such that} \\ \frac{\nu}{2} \int_0^1 \partial_x u_h v \, dx + \int_0^1 \varepsilon \partial_x u_h \partial_x v \, dx = \int_0^1 f v \, dx \quad \forall v \in V_{h,0}. \end{cases}$$

(c)



One can use the hat functions $\varphi_j(x)$ such that:

$$\varphi_j(x_i) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

as a basis set of $V_{h,g}$. Therefore, $u_h \in V_{h,g}$ $\exists \{\xi_j\}_{j=0}^{N}$ such that $u_h = \sum_{j=0}^{N} \xi_j \varphi_j = \xi_0 \varphi_0 + \sum_{j=1}^{N} \xi_j \varphi_j$

$$= \varphi_0 + \sum_{j=1}^{N} \xi_j \varphi_j$$

Insert it into the (GFEM) formulation and get:

$$A\bar{\xi} = \bar{b}, \text{ where}$$

$$A_{ij} = \frac{v}{2}\int_0^1 \partial_x \varphi_j \varphi_i \, dx + \int_0^1 \varepsilon \, \partial_x \varphi_j \, \partial_x \varphi_i \, dx,$$

$$b_1 = \int_0^1 f \varphi_1 \, dx - \frac{v}{2}\int_0^1 \partial_x \varphi_0 \varphi_1 \, dx - \int_0^1 \varepsilon \, \partial_x \varphi_0 \, \partial_x \varphi_1 \, dx,$$

$$b_i = \int_0^1 f \varphi_i \, dx, \quad i = 2, \dots, N$$

(d) Usually differential equation is given in the strong form. For example, solutions of (**) must be twice differentiable functions pointwise, i.e. $u(x) \in C^2(0,1)$.

However, the condition of pointwise differtiability in the classical sense, i.e. pointwise or strong derivatives are relaxed in the weak form. The weak form is an integral equation such that the solution of it differentiable in the weak sence. For example, the weak solutions of (**) must ~~be~~ integrable $\int_0^1 u^2 dx < \infty$, and the first derivative of it is also integrable $\int_0^1 (u')^2 dx < \infty$.

We are interested in weak forms due it is easy to construct finite dimentional subspace of weak solutions. Also weak equations are integral equations, therefore this can be easier to transform to discrete case and linear systems.

(e) FEM: + works for any complex grids, geometries;

+ can be made very high order;

− constructing the elements of the matrix can be expensive;

− does not work for advection dominated problems

FDM: + simple to implement, cheap to compute;

+ can be made high order easily;

− cannot handle complex geometries;

− uses only structured meshes, is difficult to refine locally;

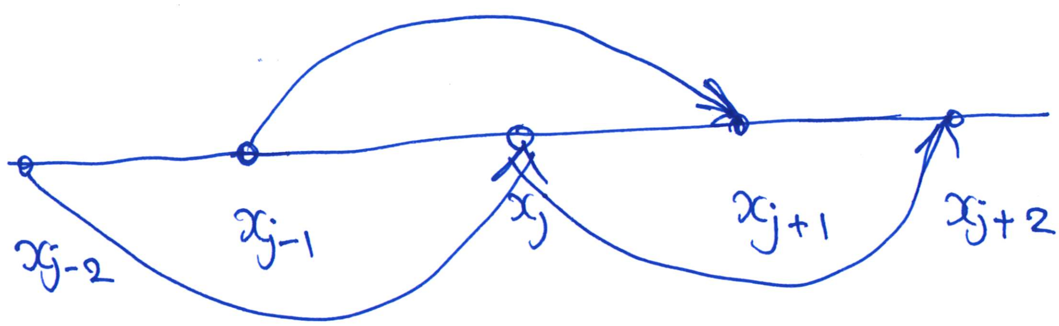
Q5. (a) The central difference approximation of (**) is

$$U \frac{u_{j+1} - u_{j-1}}{2h} - \varepsilon \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = f_j, \quad j = \overline{1, N-1}$$

When $\varepsilon \to 0$ we are left with

$$U \frac{u_{j+1} - u_{j-1}}{2h} = f_j \quad , \; j = \overline{1, N-1},$$

we see that solution at point x_{j+1} is computed using the point x_{j-1}. However it does not see any changes at x_j:



Therefore, any discontinuity at the solution will not be dumbed and produces oscillations.

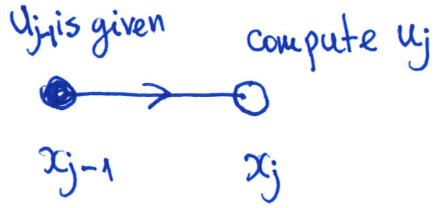
(b) The upwind scheme for $\varepsilon = 0$, $U > 0$ is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + U\,\frac{u_j^n - u_{j-1}^n}{h} = f_j \quad , \; j = \overline{1,N},$$

Since (**) is stationary it becomes:

$$U\,\frac{u_j - u_{j-1}}{h} = f_j \quad , \; j = 1, 2, \dots, N.$$

and the stencil for this case is : start with $j=1$ and



$j = 1, 2, \dots, N$

We obtain this scheme if we choose

$$\varepsilon = \frac{U h}{2}.$$

$$U\,\frac{u_{j+1} - u_{j-1}}{2h} - \frac{U h}{2}\,\frac{u_{j+1} - 2u_j + u_{j-1}}{\cancel{2}h^2} = f_j$$

$$\frac{U}{2h}\left(\cancel{u_{j+1}} - u_{j-1} - \cancel{u_{j+1}} + 2u_j - u_{j-1}\right) = f_j$$

$$\frac{U}{h}\left(u_j - u_{j-1}\right) = f_j .$$

(c) The upwind scheme for $\varepsilon=0$ and $U<0$ is

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} + U\,\frac{u_{j+1}^n-u_j^n}{h} = f_j\,, \quad j=\overline{0,N-1}\,.$$

Again, since (**) is stationary we get:

$$U\,\frac{u_{j+1}-u_j}{h} = f_j\,, \quad j=\overline{0,N-1}$$

And the stencil is

$$\underset{\text{compute } u_j}{\circ} \longleftarrow \underset{\text{given } u_{j+1}}{\bullet} \qquad j=N, N-1, \dots, 0$$

We obtain this scheme if we choose

$$\varepsilon = -\frac{Uh}{2}\,.$$

Infact:

$$U\,\frac{u_{j+1}-u_{j-1}}{2h} + \frac{Uh}{2}\,\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = f_j\,, $$

$$\frac{U}{2h}\left(u_{j+1}-\cancel{u_{j-1}}+u_{j+1}-2u_j+\cancel{u_{j-1}}\right)=f_j$$

$$\frac{U}{h}\left(u_{j+1}-u_j\right)=f_j\,.$$