Uppsala Universitet
Institutionen för Informationsteknologi
Avdelningen för Beräkningsvetenskap

Exam – Scientific Computing 3, 2020-03-13

Time: $8^{00} - 13^{00}$
Allowed resources: Calculator, Beta Mathematics Handbook

Start a new solution on a new sheet of paper. To get full credit for your solution good arguments for your method of solution and detailed calculations are required. The total credit sum determines the grade.

To pass the exam: You need to get at least 4 points from each of the three topics Linear Systems, Finite Difference Method, Finite Element Method. Then, the following grade limits will be applied: grade 3 at least 12 points, grade 4 at least 18 points, grade 5 at least 24 points.

1. For the linear system of equations $Ax = b$ with

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 4 & 2 \\ 2 & 0 & 10 \end{pmatrix}, \quad b = \begin{pmatrix} -12 \\ 16 \\ 12 \end{pmatrix}, \tag{1}$$

   (a) LU-factorize the matrix A. (2p)

   (b) Solve the linear system using the resulting matrices L and U from the LU factorization above. (2p)

2. Analyze if the linear system in Problem 1 can be solved by the Jacobi method. If yes, use the Jacobi method to compute $x^{(1)}$ and $x^{(2)}$ for the case $x^{(0)} = [0, 0, 0]^T$. (4p)

3. Consider the following PDE in $I = [0, 1]$,

$$\partial_t u + 2020\, \partial_x u = 0, \quad (x, t) \in I \times (0, T], \tag{*}$$

   with initial condition $u(x, 0) = u_0(x)$, where u_0 is a given function, $T > 0$ is the final time. To solve this equation, you are given two finite difference schemes to choose from,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + 2020 \frac{-u_{j-1}^{n+1} + u_{j+1}^{n+1}}{2\Delta x} = 0, \tag{2}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + 2020 \frac{u_{j-2}^n - 4u_{j-1}^n + 3u_j^n}{2\Delta x} = 0. \tag{3}$$

   (a) Derive the characteristic lines of (*) and sketch them in the x-t-plane. On what boundaries do we need to set boundary conditions for the equation? (2p)

   (b) Derive the local trunction error of (2) and (3). What are the orders of convergence in time and space? (4p)

   (c) Consider (*) as a periodic problem. Analyze stability of the method (2) using Von Neumann analysis. Which conclusions can you draw about convergence of (2) in that case? (2p)

(d) Consider also the periodic problem. Assuming that there exists a constant $C < 1$ such that under $\Delta t \leq C\Delta x$, (3) is stable. What are the advantages and the disadvantages of (2) and (3) when solving (*)? (2p)

4. Consider the following stationary advection-diffusion equation:

$$\begin{cases} \partial_x \frac{Uu}{2} - \partial_x(\varepsilon \partial_x u) &= f(x), \quad 0 < x < 1, \\ u(0) &= 1, \\ u'(1) &= 0, \end{cases} \tag{**}$$

where $U > 0$, $\varepsilon > 0$ are constants and $f(x)$ is a given source function.

(a) Formulate the weak formulation with appropriate function spaces. (1.5p)

(b) Discretize the weak formulation using the finite element method in continuous piecewise linear polynomial spaces at the nodes $0 = x_0 < x_1 < \ldots < x_N = 1$, where N is a positive integer. (1.5p)

(c) Construct the final linear system to be solved. Write down the elements of the matrix and how the boundary terms are included in the right-hand side vector. You do not need to compute the integrals. (2p)

(d) What is the difference between strong and weak forms? And why are we interested in weak formulations? (1p)

(e) List at least two advantages and two disadvantages for each finite difference and finite element methods. (1p)

5. Consider again the stationary advection-diffusion equation (**). If we discretize this equation using continuous piecewise linear finite elements or second order finite differences, we get equivalent schemes.

(a) Write down the central difference approximation of the equation on the points x_{j-1}, x_j, x_{j+1}. Explain what happens to the solution when ε goes to zero. (2p)

(b) Suggest a value for ε such that the scheme becomes an upwind type scheme. What would be the stencil for this scheme? Draw the stencil. Show the details of your computation. (1.5p)

(c) Let us assume that U in (**) is a negative constant. What would be the value of ε for this case and what would be the corresponding upwind type scheme? What would be the stencil for this scheme? Draw the stencil. (1.5p)

Good Luck!