Exam 2019-10-28

1) a) $A = \begin{pmatrix} 10 & 9 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & 6 \end{pmatrix}$

LU-factorize A

$$\overset{+1/10}{\llcorner\!\!\rightarrow} \begin{pmatrix} 10 & 9 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & 6 \end{pmatrix} \Rightarrow \overset{\frac{9/10}{2}}{\llcorner\!\!\rightarrow} \begin{pmatrix} 10 & 9 & 1 \\ 0 & -2 & 1 \\ 0 & 9/10 & \frac{61}{10} \end{pmatrix} \Rightarrow \begin{pmatrix} 10 & 9 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 131/20 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/10 & -9/20 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 10 & 9 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 131/20 \end{pmatrix}$$

b) Solve $Ax = b$ $\quad b = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\Leftrightarrow$

$LUx = b$

$\Leftrightarrow$

1) $Lz = b$

2) $Ux = z$

1) $z_1 = 1$

$z_2 = 2$

$$-\frac{z_1}{10} - \frac{9z_2}{20} + z_3 = -1 \qquad z_3 = -1 + \frac{1}{10} + \frac{9 \cdot 2}{20} = 0$$

2)
$$\begin{aligned} 10x_1 + 9x_2 + x_3 &= 1 \\ -2x_2 + x_3 &= 2 \\ \frac{131}{20} x_3 &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x_3 &= 0 \\ x_2 &= -1 \\ x_1 &= \frac{1 - 9x_1}{10} = 1 \end{aligned}$$

2) Analyze Jacobi for

$$A = \begin{pmatrix} 10 & 9 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & 6 \end{pmatrix}$$

Convergence if $|a_{ii}| > \sum_{j \neq i}^{n} |a_{ij}|$ all i
Not fullfilled for $i = 1$

form: $G_J = I - D^{-1} A = \begin{pmatrix} 0 & -9/10 & -1/10 \\ 0 & 0 & +1/2 \\ -1/6 & 0 & 0 \end{pmatrix}$

Convergence if $\|G_J\| < 1$ any norm

$\|G_J\|_1 = \max\left(1/6 , 9/10 , 1/10 + 1/2 \right) = 9/10 < 1$

$\Rightarrow$ Jacobi converges

3) $u_t = \lambda u_x \qquad \lambda > 0$

1) $\dfrac{u_j^{n+1} - u_j^n}{\Delta t} = \lambda \left(\dfrac{u_{j+1}^n - u_j^n}{\Delta x} \right)$

2) $\dfrac{u_j^{n+1} - u_j^n}{\Delta t} = \lambda \left(\dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right)$

1) $\tau(\Delta x, \Delta t) = \dfrac{u_j^{n+1} - u_j^n}{\Delta t} - \lambda \left(\dfrac{u_{j+1}^n - u_j^n}{\Delta x} \right) = \left[\begin{array}{l} \text{Taylor} \\ \text{around } x_j, t_n \end{array} \right]$

$= \dfrac{u + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + O(\Delta t^3) - u}{\Delta t} - \lambda \left(\dfrac{u + \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} + O(\Delta x^3) - u}{\Delta x} \right)$

$= u_t - \lambda u_x + \dfrac{\Delta t}{2} u_{tt} + O(\Delta t^2) - \dfrac{\lambda \Delta x}{2} u_{xx} + O(\Delta x^2)$

$\Rightarrow$ Order $(1, 1)$

2) $\tau(\Delta x, \Delta t) = \dfrac{u_j^{n+1} - u_j^n}{\Delta t} - \lambda \left(\dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right) =$

$= u_t + \dfrac{\Delta t}{2} u_{tt} + O(\Delta t^2) - \lambda \Big(u + \Delta x u_x + \dfrac{\Delta x^2}{2} u_{xx} + \dfrac{\Delta x^3}{6} u_{xxx} + O(\Delta x^4)$

$- \left(u - \Delta x u_x + \dfrac{\Delta x^2}{2} u_{xx} - \dfrac{\Delta x^3}{6} u_{xxx} + O(\Delta x^4) \right) \Big) / 2\Delta x$

$= u_t + \dfrac{\Delta t}{2} u_{tt} + O(\Delta t^2) - \lambda u_x + \lambda \dfrac{\Delta x^2}{6} u_{xxx} + O(\Delta x^3)$

$\Rightarrow$ Order $(1, 2)$

4) Stability condition

(2) $u_j^n = q^n e^{iwx_j}$

$\Rightarrow (q-1)q^n e^{iwx_j} = \lambda \cdot \frac{\Delta t}{\Delta x} q^n e^{iwx_j} \left(\frac{e^{iw\Delta x} - e^{-iw\Delta x}}{2}\right)$

$\Rightarrow q = 1 + \frac{\lambda \Delta t}{\Delta x} \left(\frac{\cos(w\Delta x) + i\sin(w\Delta x) - (\cos(w\Delta x) - i\sin(w\Delta x))}{2}\right)$

$= 1 + \frac{\lambda \Delta t}{\Delta x} i\sin(w\Delta x)$

$|q| > 1$ for some w, Unstable!

(1) $(q-1)q^n e^{iwx_j} = \frac{\lambda \Delta t}{\Delta x} q^n e^{iwx_j} (e^{iw\Delta x} - 1)$

$\Rightarrow q = 1 + \frac{\lambda \Delta t}{\Delta x} (\cos(w\Delta x) - 1 + i\sin(w\Delta x))$

$|q|^2 = \left(1 + \frac{\lambda \Delta t}{\Delta x}(\cos(w\Delta x) - 1)\right)^2 + \left(\frac{\lambda \Delta t}{\Delta x}\sin(w\Delta x)\right)^2$

Let $\alpha = w\Delta x$, $\beta = \frac{\lambda \Delta t}{\Delta x}$

show first stability for $\underline{\beta = 1}$

$|q|^2 = (1 + (\cos(\alpha) - 1))^2 + \sin^2(\alpha) =$

$= 1 + 2\cos(\alpha) - 2 + \cos^2\alpha - 2\cos\alpha + 1 + \sin^2\alpha$

$= \cos^2\alpha + \sin^2\alpha = 1$ stable!

$\underline{\beta < 1}$

$$|q|^2 = (1 + \beta(\cos(\alpha) - 1))^2 + \beta^2 \sin^2\alpha$$
$$= 1 + 2\beta(\cos(\alpha) - 1) + \beta^2(\cos^2\alpha - 2\cos(\alpha) + 1)$$
$$+ \beta^2 \sin^2\alpha = 1 + 2\beta(\cos(\alpha) - 1) + 2\beta^2 - 2\beta^2\cos(\alpha)$$
$$= 1 + 2\beta^2 - 2\beta\cos(\alpha)(\beta - 1) - 2\beta$$

Worst case $\cos(\alpha) = +1$

$$|q|^2 = 1 + 2\beta^2 - 2\beta^2 + 2\beta - 2\beta = 1$$

for other α $|q|^2 < 1$

$\underline{\beta > 1}$

$$|q|^2 = 1 + 2\beta^2 - 2\beta\cos(\alpha)(\beta - 1) - 2\beta$$

Worst case $\cos(\alpha) = -1$

$$\Rightarrow |q|^2 = 1 + 2\beta^2 + 2\beta^2 - 2\beta - 2\beta = 1 + 4\beta(\beta - 1) > 1$$

5) $\begin{cases} u'' + qu = f(x) & 0 \le x \le 1 \\ u(0) = 0 \\ u'(1) = \beta \end{cases}$

1) Discretize $\{x_i\}_{i=0}^{N}$ $x_i = i \cdot h$ $h = 1/N$

2) Define $\{\phi_i\}_{i=1}^{N}$ Hat-functions

3) Approximate $u(x) \approx \tilde{u}(x) = \sum_{j=1}^{N} c_j \phi_j(x)$

4) Form $r(\tilde{u}) = \tilde{u}'' + q\tilde{u} - f$

5) Req $(r(\tilde{u}), \phi_i) = 0$ $i = 1, \dots, N$

$$\Longleftrightarrow \int_0^1 \tilde{u}'' \phi_i \, dx + \int_0^1 q \tilde{u} \phi_i \, dx = \int_0^1 f(x) \phi_i \, dx \quad i = 1, \dots, N$$

6) Integrate by parts

$$\int_0^1 \tilde{u}'' \phi_i \, dx = \underbrace{[\tilde{u}' \cdot \phi_i]_0^1}_{\beta \phi_i(1)} - \int_0^1 \tilde{u}' \phi_i' \, dx \quad i = 1, \dots, N$$

7) Form $Ac = b$

$(r(\tilde{u}), \phi_i) = 0$ $i = 1, \dots, N$

$$\Longleftrightarrow$$

$$-\int_0^1 \sum_{j=1}^{N} c_j \phi_j' \phi_i' \, dx + \int_0^1 q \sum_{j=1}^{N} c_j \phi_j \phi_i \, dx = \int_0^1 f(x) \phi_i \, dx - \beta \phi_i(1)$$

$$\Longleftrightarrow$$

$$\sum_{j=1}^{N} c_j \left(-\int_0^1 \phi_j' \phi_i' \, dx + q \int_0^1 \phi_j \phi_i \, dx \right) = \int_0^1 f(x) \phi_i \, dx - \beta \phi_i(1) \quad i = 1, \dots, N$$

Mathematics Handbook

$$\int_0^1 \phi_j' \phi_i' \, dx = \begin{cases} 2/h & i=j\neq N \\ -1/h & |i-j|=1 \\ 0 & |i-j|>1 \end{cases}$$

$i=j=N$

$$\int_0^1 \phi_N' \phi_N' \, dx = \int_{x_{N-1}}^{x_N} (1/h)^2 \, dx = 1/h$$

$$\int_0^1 \phi_j \phi_i \, dx = \begin{cases} \frac{2h}{3} & i=j\neq N \\ h/6 & |i-j|=1 \\ 0 & |i-j|>1 \end{cases}$$

$i=j=N$

$$\int_0^1 \phi_N \cdot \phi_N \, dx = \int_{x_{N-1}}^{x_N} \phi_N^2 \, dx = \int_0^h \left(\frac{\tilde{x}}{h}\right)^2 d\tilde{x} = \left[\frac{\tilde{x}^3}{3h^2}\right]_0^h = \frac{h}{3}$$

$$\Rightarrow A = \begin{pmatrix} (-2/h + q\cdot 2h/3) & (+1/h + qh/6) & & \\ (+1/h + qh/6) & \ddots & \ddots & \\ & \ddots & \ddots & \\ & & & (-1/h + q\frac{h}{3}) \end{pmatrix}$$

$$b = \begin{bmatrix} f(x_1)\cdot h \\ \vdots \\ \vdots \\ \frac{f(x_N)}{2} h - \beta \end{bmatrix}$$

$$b_i = \int_0^1 f(x)\phi_i(x)\,dx - \beta \phi_i(1) \quad \text{[Trapets]}$$

$$\approx \begin{cases} f(x_i)h & i\neq N \\ \frac{f(x_i)h}{2} - \beta & i=N \end{cases}$$

6) $\begin{cases} u'' = f(x) & 0 \le x \le 1 \\ u'(0) = \alpha - u(0) \\ u(1) = 0 \end{cases}$

1) Discretize $\{x_i\}_{i=0}^{N}$ $x_i = ih$ $h = 1/N$

2) Define $\{\phi_i\}_{i=0}^{N-1}$ Hat-functions

3) Approximate $u(x) \approx \tilde{u}(x) = \sum_{j=0}^{N-1} c_j \phi_j(x)$

4) Form $r(\tilde{u}) = \tilde{u}'' - f$

5) Require $(r(\tilde{u}), \phi_i) = 0$ $i = 0, .., N-1$

$\iff$

$$\int_0^1 \tilde{u}'' \phi_i \, dx = \int_0^1 f(x) \cdot \phi_i \, dx \qquad i = 0, .., N-1$$

6) Integrate by parts

$$\int_0^1 \tilde{u}'' \phi_i \, dx = \underbrace{[\tilde{u}' \phi_i]_0^1}_{-(\alpha - \tilde{u}(0)) \cdot \phi_i(0)} - \int_0^1 \tilde{u}' \phi_i' \, dx = \tilde{u}(0) - \alpha - \int_0^1 \tilde{u}' \phi_i' \, dx$$

7) Form $Ac = b$

$(r(\tilde{u}), \phi_i) = 0$ $i = 0, .., N-1$

$\iff$

$$\phi_i(0) \cdot \left(\sum_{j=0}^{N-1} c_j \phi_j(0) - \alpha \right) - \int_0^1 \sum_{j=0}^{N-1} c_j \phi_j' \phi_i' \, dx = \int_0^1 f(x) \phi_i \, dx \qquad i = 0, .., N-1$$

$\iff$

$$c_0 \phi_i(0) + \sum_{j=0}^{N-1} c_j \left(-\int_0^1 \phi_j' \phi_i' \, dx\right) = \int_0^1 f(x)\phi_i \, dx + \alpha \, \phi_i(0)$$

$$\iff Ac = b$$

Half-base function

$$\int_0^1 \phi_0' \phi_0' \, dx = 1/h$$

$$\int_0^1 \phi_j' \phi_i' \, dx = \begin{cases} 2/h & i = j \neq 0 \\ -1/h & |i-j| = 1 \\ 0 & |i-j| > 1 \end{cases}$$

$$b_i = \begin{cases} f(x_i) \cdot h & i \neq 0 \\ \frac{f(x_0) \cdot h}{2} + \alpha & i = 0 \end{cases}$$

$$A = \begin{pmatrix} 1 - 1/h & +1/h & & \\ 1/h & -2/h & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix} \qquad b = \begin{pmatrix} \frac{f(x_0) \cdot h}{2} + \alpha \\ f(x_1) \cdot h \\ \vdots \\ f(x_{N-1}) \cdot h \end{pmatrix}$$