Uppsala University
Department of Information technology

Examination in Scientific computing 3, 2019-10-28

Time: 08.00 – 13.00
Allowed resources: Beta Mathematics Handbook

Start a new solution on a new sheet of paper and write your code on every sheet.

Each problem can give up to 5 credits. To get full credit for your solution good arguments for your method of solution and detailed calculations are required. The total credit sum determines the grade.

Grade limits: You need to solve at least one problem from each of the three topics satisfactory, i.e., you need to get at least 3 points for one of the problems in Linear Systems (1 and 2), FDM (3 and 4) and FEM (5 and 6). Then for grade 3 you need at least 12 points in total, grade 4 at least 18 points, grade 5 at least 24 points.

1. For the linear system of equations $Ax = b$ with

$$A = \begin{pmatrix} 10 & 9 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & 6 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

   a) LU-factorize the matrix A (without pivoting).
   b) Solve the linear system using the resulting matrices L and U from your factorization above.

2. Analyze if we can use Jacobi's iterative method to solve the linear system in Problem 1 above.

3. For the PDE

$$u_t = \lambda u_x, \quad \lambda > 0$$

   we are considering to use one the two Finite Difference Methods below

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \lambda \left(\frac{u_{j+1}^n - u_j^n}{\Delta x} \right) \tag{1}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \lambda \left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right) \tag{2}$$

   Derive the order of accuracy for both methods.

4. Derive stability conditions for the two methods above using the Fourier method (von Neumann analysis). What are your conclusions, which method should we choose?
   Hint: One of the methods is unstable for all Δt and the other is stable for the CFL condition $\lambda \frac{\Delta t}{\Delta x} \leq 1$. Show first that the method is stable for $\lambda \frac{\Delta t}{\Delta x} = 1$, then show that it is also stable for $\lambda \frac{\Delta t}{\Delta x} < 1$ and unstable for $\lambda \frac{\Delta t}{\Delta x} > 1$.

5. Formulate the FEM for the boundary value problem

$$
\begin{aligned}
&u'' + q \cdot u = f(x), \quad 0 \leq x \leq 1 \\
&u(0) = 0 \\
&u'(1) = \beta
\end{aligned}
$$

   using piecewise linear basis functions. Derive the resulting linear system of equations. You may assume equal step length $x_{i+1} - x_i = h$ between all nodes.

6. In the course we have so far only used pure Dirichlet and/or Neumann boundary conditions, in this problem we have a mixed type boundary condition.

$$
\begin{aligned}
&u'' = f(x), \quad 0 \leq x \leq 1 \\
&u'(0) = \alpha - u(0) \\
&u(1) = 0
\end{aligned}
$$

   Formulate the FEM for the boundary value problem using piecewise linear basis functions. Derive the resulting linear system of equations. You may assume equal step length $x_{i+1} - x_i = h$ between all nodes.

Good Luck!