Brief solutions to exam 2018-10-22

1) a) $x^{(1)} = \begin{bmatrix} 0 \\ -0{,}125 \\ 0{,}5 \end{bmatrix}$

b) Strictly diagonally dominant $A \Rightarrow$ convergent

c) Not symmetric, not pos. def $x^T A x \not> 0$ $\left(x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$
$\Rightarrow$ CG not applicable

2) For the first step use

$$\frac{u_j^1 - u_j^0}{\Delta t} = \lambda D_+ D_- u_j^0 \qquad \text{(one step method)}$$

or Taylor in time

$$u_j^1 = u_j^0 + \Delta t \cdot u_t(x_j, 0) + O(\Delta t^2)$$

$$u_t(x_j, 0) = \lambda u_{xx}(x_j, 0) = \lambda f''(x_j)$$

$$\Rightarrow u_j^1 = f(x_j) + \Delta t \cdot \lambda f''(x_j)$$

Code:

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Δx = 1/N
Δt = ...
x = 0 : Δx : 1 ;
u = f(x)
unew = f(x) + Δt·λ f''(x)
for t = 2*Δt : Δt : T
   uold = u;
   u = unew;
   unew(2:end-1) = uold(2:end-1) + 2Δt·λ/Δx² (u(3:end) - 2u(2:end-1)
                                              + u(1:end-2))
   u(1) = α
   u(end) = β
end
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3) $\tau(\Delta t, \Delta x) = \frac{\Delta t^2}{6} u_{ttt} - \lambda \frac{\Delta x^2}{12} u_{xxxx}$ + higher order terms

order (2,2)

4) $u_j^n = q^n e^{i w x_j}$

$$\Rightarrow \frac{(q^2-1)}{2\Delta t} q^{n-1} e^{i w x_j} = \lambda q \cdot q^{n-1} \underbrace{D_+ D_- e^{i w x_j}}_{-\frac{4\sin^2\left(\frac{w\Delta x}{2}\right)}{\Delta x^2} \cdot e^{i w x_j}}$$

$$\Rightarrow q^2 - 1 = -\frac{8\lambda \Delta t}{\Delta x^2} \sin^2\left(\frac{w \Delta x}{2}\right) \cdot q$$

$$\Rightarrow q = -\frac{4\lambda \Delta t}{\Delta x^2} \sin^2\left(\frac{w\Delta x}{2}\right) \pm \underbrace{\sqrt{\left(\frac{4\lambda\Delta t}{\Delta x^2}\sin^2\left(\frac{w\Delta x}{2}\right)\right)^2 + 1}}_{>1}$$

One root $|q(w)| > 1$

Unstable method, can not be used for this problem!

5) $u'' = f(x) \quad 0 \le x \le 1$

$u(0) = \alpha \qquad h = 1/N$

$u'(1) = \beta$

$\tilde{u} = \sum_{j=0}^{N} c_j \phi_j(x) \qquad \phi_j(x)$ hat-functions

$r(\tilde{u}) = \tilde{u}'' - f(x) \qquad \tilde{u}(0) = \alpha \Rightarrow c_0 = \alpha$

Req $(r(\tilde{u}), \phi_i) = 0 \quad i = 1, \dots, N$ (exclude $i=0$ as c_0 is known)

$$\int_0^1 \tilde{u}'' \phi_i \, dx = [\tilde{u}' \phi_i]_0^1 - \int_0^1 \tilde{u}' \phi_i' \, dx \qquad \begin{pmatrix} \phi_i(0) = 0 \\ i = 1, \dots, N \end{pmatrix}$$

$$= \tilde{u}'(1) \cdot \phi_i(1) - \int_0^1 \tilde{u}' \phi_i' \, dx$$

$$= \beta \phi_i(1) - \int_0^1 \tilde{u}' \phi_i' \, dx$$

$$\Rightarrow (r(\tilde{u}), \phi_i) = \beta \phi_i(1) - \int_0^1 \tilde{u}' \phi_i' \, dx - \int_0^1 f(x) \cdot \phi_i \, dx$$

$\Leftrightarrow$

$$-\int_0^1 \sum_{j=0}^{N} c_j \phi_j' \phi_i' \, dx = \int_0^1 f(x) \phi_i \, dx - \beta \phi_i(1) \qquad i = 1, \dots, N$$

But $c_0 = \alpha$ (move to RHS)

$$\Leftrightarrow \quad -\sum_{j=1}^{N} c_j \int_0^1 \phi_j' \phi_i' \, dx = \int_0^1 f(x) \phi_i \, dx - \beta \phi_i(1) + \alpha \int_0^1 \phi_0' \phi_i' \, dx$$

$i = 1, \dots, N$

$\Leftrightarrow$

$Ac = b$ where $c = \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix}$

Note ϕ_N - is only a half-base function

$\int \phi_N' \phi_N' dx = 1/h$

$$b = \begin{bmatrix} f(x_1) \cdot h - \alpha \, 1/h \\ f(x_2) \cdot h \\ \vdots \\ f(x_{N-1}) \cdot h \\ f(x_N) \cdot \frac{h}{2} - \beta \end{bmatrix} \qquad A = \begin{bmatrix} -2/h & 1/h & & \\ 1/h & -2/h & \ddots & \\ & \ddots & \ddots & 1/h \\ 0 & & 1/h & -1/h \end{bmatrix}$$

6) $u_t + x u_x = 0$

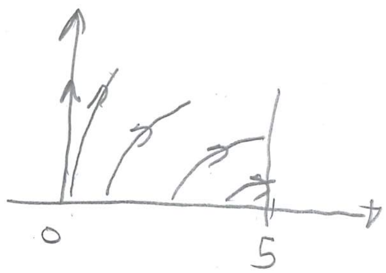
a) Char.lines $\frac{dx}{dt} = x \iff x' - x = 0$

multiply with e^{-t} and integrate

$$\Rightarrow \underbrace{e^{-t}(x' - x)}_{\frac{d}{dt} e^{-t} \cdot x(t)} = 0$$

$$\frac{d}{dt} e^{-t} \cdot x(t) = 0$$

$$\Rightarrow e^{-t} \cdot x(t) = C \iff x(t) = C \cdot e^{t}$$



At $x=0$ we have $\frac{dx}{dt} = 0$
i.e. solution is constant at $x=0$, all t

At $x=5$ we have $\frac{dx}{dt} = 5$
i.e. the solution is comming from the inside of the domain (slope $\frac{dt}{dt} = 1/5$)

B.C. $u(0,t) = u(0,0) = f(0)$ all t

No boundary cond. at $x=5$

b) Can use $\frac{u_i^{n+1} - u_i^n}{\Delta t} + x_i \left(\frac{u_i^n - u_{i-1}^n}{\Delta x} \right) = 0$ (UPW)

Taking data from the direction of char.lines (flow)
1:st order in time and space

c) CFL-condition

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{5}$$ (worst case at right boundary)