Quasi-Toeplitz matrices: analysis, algorithms and applications

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Let $a(z) = \sum_{i=-\infty}^{+\infty} a_i z^i$ be a complex valued function defined for $z \in \mathbb{C}$, $|z| = 1$. The semi-infinite matrix $T(a) = (t_{i,j})_{i,j\in\mathbb{Z}^+}$ is said *Toeplitz matrix* associated with $a(z)$ if $t_{i,j} = a_{j-i}$. Typically, Toeplitz matrices are encountered in mathematical models where a shift invariance property, in time or in space, is satisfied by some entity.

Many queueing models from the applications are described by *quasi-Toeplitz* (QT) matrices, that is, matrices of the form $A = T(a) + E$ where E is a compact correction and $a(z)$ is such that $\sum_{i=-\infty}^{+\infty} |a_i| < \infty$. For instance, in the random walk along a half-line, the probability transition matrix is the sum of a semi-infinite tridiagonal Toeplitz matrix and a correction E which is nonzero only in the entry (1,1). More complex situations are encountered if the domain of the random walk is the quarter plane where the probability matrix is block Toeplitz with Toeplitz blocks, plus finite rank corrections.

The main computational problems that one encounters in this framework include computing matrix functions, solving polynomial matrix equations, and solving linear systems where the input matrices are QT matrices.

In this talk, after pointing out the role of QT matrices in certain applications, we introduce some matrix norms, which make the class of QT matrices a Banach algebra and at the same time, are computationally tractable. Then we introduce the class of QT matrices representable by a finite number of parameters together with a matrix arithmetic on this class. This way, we may approximate QT matrices by using a finite number of parameters in the same way as real numbers are approximated by floating point numbers.

We introduce algorithms for the solution of the main computational problems encountered in this framework like computing the inverse matrix by means of the Wiener-Hopf factorization, computing the matrix exponential, solving quadratic matrix equations encountered in Quasi–Birth-Death stochastic processes where matrix coefficients are QT matrices. Examples of applications are shown together with numerical experiments.

Finally, we present some open issues and current research topics.

References

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