(A)

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FFT

$n = 2^j$

$\frac{3}{2} n \log_2 n - \frac{n}{2}$

→

COMPUTATION WITH CIRCULANTS

$n = 2^j$

- $\frac{9}{2} n \log_2 n + O(n)$
- SOLUTION OF LINEAR SYSTEMS
- MATRIX VECTOR PRODUCT

→

TOEPLITZ MATRIX-VECTOR PRODUCT

$18 n \log_2 n + O(n)$

n IS GENERIC

COMPUTATION WITH CIRCULANTS n GENERIC

FFT

$18 n \log_2 n + O(n)$

n GENERIC

B

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WE TAKE A GENERIC TOEPLITZ MATRIX OF SIZE m. THE IDEA IS TO "EMBED" SUCH A TOEPLITZ MATRIX AS PRINCIPAL LEADING SUBMATRIX OF A LARGER CIRCULANT, WHOSE SIZE IS A POWER OF 2

$$C_{2m-1} = \begin{array}{c} T \\ \left[\begin{array}{cccc|c} t_0 & t_{-1} & \cdots & t_{1-m} & \\ t_1 & \ddots & \ddots & \vdots & \\ \vdots & \ddots & \ddots & t_{-1} & \\ t_{m-1} & \cdots & t_1 & t_0 & \\ \hline t_{1-m} & & & & \\ \vdots & & & & \\ t_{-2} & & & & \\ t_{-1} & & & & \end{array}\right] \end{array}$$

PROBLEM: THE SIZE OF THE CIRCULANT IS ODD, $2m-1$; IT CANNOT BE A POWER OF 2

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$$C_m = \left[\begin{array}{cccc|ccc}
t_0 & t_{-1} & \cdots & t_{1-m} & 0 & 0 & \cdots \\
t_1 & \ddots & \ddots & \vdots & & & \\
\vdots & \ddots & \ddots & t_{-1} & & & \\
t_{m-1} & \cdots & t_1 & t_0 & & & \\
\hline
0 & & & & & & \\
0 & & & & & & \\
t_{1-m} & & & & & & \\
\vdots & & & & & & \\
t_{-2} & & & & & & \\
t_{-1} & & & & & &
\end{array}\right]$$

The upper-left block is labelled T.

$\underline{\underline{n > 2m-1}}$

* $n = 2^d \Rightarrow$

$n = 2m > 2m-1$
(ONLY ONE ZERO ADDED)

* $m = 2^d + 1$

$\underline{n = 2^{d+1}}$ IS NOT OK
BECAUSE $2^{d+1} = 2m-2$

$n = 2^{d+2} = 4m-4$
(WORST CASE)

$C_m y$ THE COMPLEXITY IS BOUNDED

$\frac{9}{2} n \log_2(n) + O(n)$

$\leq 18\, m \log_2 m + O(m)$

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(D) WE WANT TO COMPUTE $T x$

$x \in \mathbb{C}^n$, GENERIC VECTOR.

THEN WE CONSIDER $y_x \in \mathbb{C}^m$

$$y_x = \begin{bmatrix} \left.x\right\} n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \; ; \quad C_m \, y_x = \begin{bmatrix} \left.T x\right\} \\ \vdots \end{bmatrix}$$

BUT THE COMPUTATION OF $C_m y_x$ COSTS AT MOST $18 m \log_2 m + O(m)$ AND FROM $C_m y_x$, EXTRACTING THE FIRST n COMPONENTS, YOU RECOVER $T x$

(E) HOW TO HAVE A FAST FOURIER TRANSFORM FROM A FAST TOEPLITZ VECTOR PRODUCT FOR A GENERIC SIZE n?

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PRELIMINARY OBSERVATION I

TAKE X SQUARE MATRIX, $D^{(1)}$ DIAGONAL MATRIX, $D^{(2)}$ DIAGONAL MATRIX

$$\left(D^{(1)}X\right)_{J,K} = D^{(1)}_{JJ}\, X_{J,K}$$

$$\left(X D^{(2)}\right)_{J,K} = D^{(2)}_{KK}\, X_{JK}$$

$$\sqrt{n}\, F_n = \left(e^{-i\frac{2\pi JK}{n}} \right)_{J,K=0}^{n-1}$$

n GENERIC

F

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$$\sqrt{m}\, F_m = \left(e^{-i\frac{2\pi jk}{m}} \right)_{j,k=0}^{m-1} =$$

$$= \left(e^{+i\frac{\pi}{m}(-2jk)} \right)_{j,k=0}^{m-1}$$

$$-2st = (s-t)^2 - s^2 - t^2$$

PRELIMINARY OBSERVATION

$$= \left(e^{+i\frac{\pi}{m}\left((j-k)^2 - j^2 - k^2\right)} \right)_{j,k=0}^{m-1} =$$

$$= \underbrace{\operatorname{DIAG}_{0\le j\le m-1}\left(e^{-i\frac{\pi}{m}j^2}\right)}_{D} \underbrace{\left(e^{i\frac{\pi}{m}(j-k)^2}\right)}_{T} \overbrace{\operatorname{DIAG}_{0\le k\le m-1}\left(e^{-i\frac{\pi}{m}k^2}\right)}^{D}$$

COST OF $\sqrt{m}F_m x$ IS m + THE COST OF T TIMES A VECTOR

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IN CONCLUSION

$y = V_m F_m x$ CAN BE COMPUTED AS

$$Y_1 = D x$$
$$Y_2 = T Y_1$$
$$Y_3 = D Y_2$$
$$Y = V_m Y_3$$

WITHIN $18 m \log_2 m + O(m)$ OPERATIONS

NOW WE ANALYZE MORE IN DETAIL TOEPLITZ STRUCTURES, FROM A SPECTRAL VIEW POINT WITH THE IDEA OF HAVING TOOLS FOR ANALYZING THE PERFORMANCE OF GIVEN PRECONDITIONERS

H

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- Given $f \in L^1(-\pi,\pi)$ we have defined

$$T_n(f) = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{1-n} \\ a_1 & \ddots & & \vdots \\ \vdots & & \ddots & a_{-1} \\ a_{n-1} & \cdots & a_1 & a_0 \end{pmatrix}, \quad a_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-ij\theta}\, d\theta$$

- We will show a lot of relationships between the spectral properties of $T_n(f)$ and f itself (generating function)

<u>Fact 1:</u> f <u>real-valued</u> almost everywhere

$$\Rightarrow \overline{a_j} = \frac{1}{2\pi}\overline{\int_{-\pi}^{\pi} f(\theta) e^{-ij\theta}\, d\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(\theta)}\, e^{ij\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int f(\theta) e^{+ij\theta} \underset{\text{def}}{=} a_{-j} \quad \forall j \in \mathbb{Z} \Rightarrow T_n(f) \text{ Hermitian}$$

(I)

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f REAL VALUED A.E $\Rightarrow$ $a_j(T_m(f)) \in \mathbb{R}$

($T_m(f)$ HERMITIAN)

THERE IS MORE ... IF $f \geq 0$ AE. THEN

$T_m(f)$ IS NONNEGATIVE DEFINITE AND

IN FACT IF ESS SUP $f > 0$ THEN $T_m(f)$ IS

POSITIVE DEFINITE....

- $f \geq 0$ A.E MEANS THAT $T_m(f)$ IS HERMITIAN.

$\underline{A}$ POSITIVE DEFINITE IFF $\forall v \neq 0_{\mathbb{C}^n}$ $v^* A v > 0$!

$v \neq 0_{\mathbb{C}^n}$

$$v^* T_m(f) v = \sum_{j,k=0}^{m-1} \bar{v}_j (T_m(f))_{j,k} v_k = \sum_{j,k=0}^{m-1} \bar{v}_j a_{j-k}(f) v_k$$

$$= \sum_{j,k=0}^{m-1} \bar{v}_j v_k \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-i(j-k)\theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j,k=0}^{m-1} \bar{v}_j v_k e^{-ij\theta} \cdot e^{ik\theta} f(\theta)\, d\theta$$

(L)

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$$v^* T_n(f) v = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{j=0}^{n-1} \bar{v}_j e^{-ij\theta} \right) \left(\sum_{k=0}^{n-1} v_k e^{ik\theta} \right) f(\theta)\, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_v(\theta)|^2 f(\theta)\, d\theta \qquad P_v(\theta) = \sum_{r=0}^{n-1} v_r e^{ir\theta}$$

$v \neq 0_{\mathbb{C}^n}$

$\underline{v^* T_n(f) v > 0}$ IF ess sup $f > 0$, $f \geq 0$

$v^* T_n(f) v = 0$ IF AND ONLY IF $f \geq 0$ AND $f \equiv 0$.

LOCALIZATION RESULT.

$f \in L^1(-\pi, \pi)$, $r =$ ess inf f, $R =$ ess sup f

THEN A) IF $r = R$, $T_n(f) = r I$, $\lambda_j(T_n(f)) = r$ $\forall j \; \forall n$

B) IF $r < R$, $\forall n \; \forall j$ $\lambda_j(T_n(f)) \in (r, R)$

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PROOF: A) IS TRIVIAL IF $f \equiv \tau \Rightarrow$

$$a_J(\tau) = \begin{cases} \tau & J=0 \\ 0 & \text{OTHERWISE} \end{cases} \Leftrightarrow$$

$$T_n(f) = \tau I$$

B) IF $\tau < R$: $f - \tau \geq 0$ A.E

$T_n(f-\tau)$ IS POSITIVE DEFINITE i.e.

$$\lambda_J(T_n(f-\tau)) > 0 \quad \forall J=1,\dots,n,\ \forall n$$

BUT $T_n(f-\tau) = T_n(f) - \tau I$. THEN

$$0 < \lambda_J(T_n(f-\tau)) = \lambda_J(T_n(f)) - \tau$$

$$\lambda_J(T_n(f)) > \tau$$

THE OTHER IS THE SAME

(N)

$$T_n = \begin{bmatrix} 6 & -4 & 1 & & 0 \\ -4 & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & -4 \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & -4 & 6 \end{bmatrix}$$

T_n IS A UNIFORM NORM DISCRETIZATION BY FINITE DIFF. OF

$$\left\{ \begin{array}{l} u^{(IV)} \text{ ON } (0,1) \\ u^{(j)}(0) = u^{(j)}(1) = 0, \ j = 0,1 \end{array} \right.$$

$a_0 = 6, \quad a_1 = a_{-1} = -4, \quad a_2 = a_{-2} = 1$

$$f(\theta) = 6 - 4e^{i\theta} - 4e^{-i\theta} + e^{2i\theta} + e^{-2i\theta}$$
$$= 6 - 8\cos\theta + 2\cos 2\theta = (2 - 2\cos\theta)^2$$

$\min f = 0, \quad \max f = 16$

$\Rightarrow \quad T_n(f)$ HAS EIGENVALUES IN $(0,16)$

EX:

$- T_n(\theta^2 h(\theta)) \quad , \quad h(\theta) \in L^\infty$
$1 \le h(\theta) \le 3$

$\lambda_\sigma(T_n(\theta^2 h(\theta))) \in (0, 3\pi^2)$

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AGAIN A BIT OF THEORY

$\mathcal{A}$ VECTOR SPACE WITH A PARTIAL ORDERING

$\mathcal{B}$ " " " " " "

DEFINITION

- $\Phi : \mathcal{A} \to \mathcal{B}$
- Φ IS LINEAR
- Φ IS POSITIVE THAT IS $f \geq 0_{\mathcal{A}} \Rightarrow \Phi(f) \geq 0_{\mathcal{B}}$

IN THIS CASE WE SAY THAT Φ IS LINEAR AND POSITIVE (Φ LPO)

ACCORDING TO WHAT WE HAVE PROVEN

$T_n : L^1(-\pi, \pi) \longrightarrow M_n(\mathbb{C})$ IS LPO

T_n MAPS NONNEGATIVE FUNCTIONS INTO NONNEGATIVE DEFINITE MATRICES

(P)

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EXAMPLES FROM THAT SPACES OF FUNCTION

- $\mathcal{A} = C[0,1]$ CONTINUOUS FUNCTIONS

$B = \mathcal{A}$

$$B_m(f(\cdot))(x) = \sum_{k=0}^{m} \binom{m}{k} f\left(\frac{k}{m}\right) x^k (1-x)^{m-k}$$

$B_m : C[0,1] \to C[0,1]$ IS LINEAR AND POSITIVE : <u>BERNSTEIN POLYNOMIALS</u>

T_m IS LPO

B_m IS LPO

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KOROVKIN THEOREM (1962)

- Φ_n A SEQUENCE OF LINEAR POSITIVE OPERATORS FROM $C[0,1]$ to $C[0,1]$

- $$\| \Phi_n(g(t))(x) - g(x) \|_\infty \xrightarrow[n \to \infty]{} 0, \quad g = 1,\ g = t,\ g = t^2$$

$$\Downarrow$$

$$\forall f \in C[0,1], \quad \| \Phi_n(f(t))(x) - f(x) \|_\infty \xrightarrow[n \to \infty]{} 0$$

$$\longleftrightarrow$$