A1

9-3-21

$\phi_n : C[0,1] \to C[0,1]$ LINEAR AND POSITIVE

$\underline{\alpha}$ $\phi_n(\alpha f + \beta g) = \alpha \phi_n(f) + \beta \phi_n(g)$

$\underline{\beta}$ IF $f \geq 0$ THEN $\phi_n(f) \geq 0$

- FROM THIS FACT THAT ϕ_n IS LPO, WE DEDUCE IN "ONE LINE" THAT ϕ_n IS MONOTONE: $f_1 \geq f_2$ THEN $f_1 - f_2 \geq 0$. CONSEQUENTLY BY POSITIVITY (β) $\phi_n(f_1 - f_2) \geq 0$; THEREFORE BY LINEARITY $\phi_n(f_1) - \phi_n(f_2) \geq 0$ IFF $\phi_n(f_1) \geq \phi_n(f_2)$

- AS A CONSEQUENCE, IT IS EVIDENT THAT ϕ_n IS ALSO ISOTONE: $-|f| \leq f \leq |f|$. BY MONOTONICITY $-\phi_n(|f|) \leq \phi_n(f) \leq \phi_n(|f|)$ IFF $|\phi_n(f)| \leq \phi_n(|f|)$

B1

9-3-21

$J=0,1,2$

$\rightarrow \quad \Phi_m(t^J)(x) = x^J + \mathcal{E}_m^{(J)}(x) \quad , \quad \|\mathcal{E}_m^{(J)}(x)\|_\infty \xrightarrow[m\to\infty]{} 0$

$\forall f \in C[0,1]$

$\forall \varepsilon>0, \exists m_\varepsilon \quad |\Phi_m(f(t))(x) - f(x)| < \varepsilon \quad , \quad m \geq m_\varepsilon \quad \forall x \in [0,1]$

WE WORK WITH THE QUANTITY

$$|\Phi_m(f(t))(x) - f(x)| = |\Phi_m(f(t))(x) - (\Phi_m(1)(x) - \mathcal{E}_m^{(0)}(x))f(x)|$$

$$= |\Phi_m(f(t))(x) - f(x)\,\Phi_m(1)(x) + \mathcal{E}_m^{(0)}(x)\,f(x)| \underset{\text{LINEARITY}}{=}$$

$$= |\Phi_m(f(t)-f(x))(x) + \mathcal{E}_m^{(0)}(x)\,f(x)|$$

TRIANGLE INEQ.

$$\leq |\Phi_m(f(t)-f(x))(x)| + \|\mathcal{E}_m^{(0)}(x)\|_\infty \cdot \|f\|_\infty$$

$\|\mathcal{E}_m^{(0)}(x)\|_\infty \|f\|_\infty < \varepsilon/3 \quad \bar{m}_\varepsilon \quad \forall m \geq \bar{m}_\varepsilon$

(C1)

9-3-21

$$|\Phi_m(f(t))(x) - f(x)| \leq |\Phi_m(f(t)-f(x))(x)| + \varepsilon/3$$

$m \geq \bar{m}_\varepsilon$

$$\leq \Phi_m(|f(t)-f(x)|)(x) + \varepsilon/3$$

$\downarrow$

ISOTONY

NOW WE LOOK FOR A CLEVER MAJORANT OF $|f(t)-f(x)|$.

SINCE f IS CONTINUOUS IN $[0,1]$, THEN f IS UNIFORMLY CONTINUOUS SO $\forall \eta > 0 \ \exists \delta > 0$:
$|t-x| < \delta \Rightarrow |f(t)-f(x)| < \eta$.

$$|f(t)-f(x)| \leq \overbrace{\chi_{\{|t-x|<\delta\}}(t)}^{\leq 1} \cdot \eta + \chi_{\{|t-x|\geq\delta\}}(t)\, 2\|f\|_\infty$$

$$\leq \eta + 2\|f\|_\infty \frac{(t-x)^2}{\delta^2}$$

$\left[|t-x| \geq \delta \Leftrightarrow \frac{|t-x|}{\delta} \geq 1 \Leftrightarrow \frac{|t-x|^2}{\delta^2} \geq 1 \right]$

D1

9-3-21

$$\left|\Phi_m(f(t))(x) - f(x)\right| \leq \dots \leq \quad \text{MONOTONICITY}$$

$$\Phi_m(|f(t)-f(x)|)(x) + \varepsilon/3 \leq$$

$$\leq \Phi_m\left(\eta + \frac{2\|f\|_\infty (t-x)^2}{\delta^2}\right)(x) + \varepsilon/3 =$$

$$= \eta\,\Phi_m(1) + \frac{2\|f\|_\infty}{\delta^2}\Phi_m(t^2+x^2-2tx)(x) + \varepsilon/3$$

$$\leq 2\eta + \frac{2\|f\|_\infty}{\delta^2}\left(\Phi_m(t^2)(x) + x^2\Phi_m(1)(x) - 2x\,\Phi_m(t)\right)(x) + \varepsilon/3$$

$$= 2\eta + \frac{2\|f\|_\infty}{\delta^2}\left(\cancel{x^2} + \varepsilon_m^{(2)}(x) + \cancel{x^2}(1+\varepsilon_m^{(0)}(x)) \cancel{-2x}(\cancel{x}+\varepsilon_m^{(1)}(x))\right) +$$

$$= 2\eta + \overbrace{\frac{2\|f\|_\infty}{\delta^2}\left(\varepsilon_m^{(2)}(x) + x^2\varepsilon_m^{(0)}(x) - 2x\,\varepsilon_m^{(1)}(x)\right)}^{\to 0} + \varepsilon/3$$

$$\leq 2\eta + \varepsilon/3 + \varepsilon/3 < \varepsilon \qquad \begin{array}{l} m \geq \overline{m}_\varepsilon \\ \eta = \varepsilon/6 \end{array}$$

E1

9-3-21

MATRIX ALGEBRA PRECONDITIONING.

WE CHOOSE THE PRECONDITIONER BELONGING TO A MATRIX ALGEBRA RELATED TO UNITARY TRANSFORM

$$\mathcal{A}_{U_n} = \{ X \in M_n(\mathbb{C}) : X = U_n D U_n^*, \; D \text{ DIAGONAL} \}, \quad U_n^* U_n = I$$

($U_n = F_n$: $\mathcal{A}_{F_n}$ = CIRCULANTS,
ω - CIRCULANTS, $|\omega| = 1$, & SINE TRANSFORM ALGEBRAS, & COSINE TRANSFORM ALGEBRAS, & HARTLEY TRANSFORM ALGEBRAS, WAVELET TRANSFORM ALGEBRAS - - -)

F1

9-3-21

GIVEN $A \in M_n(\mathbb{C})$, GIVEN $\mathcal{A}_{U_n}$ OUR MATRIX ALGEBRA, AS PRECONDITIONER FOR A WE CHOOSE

$$P_{U_n}(A) = \underset{X \in \mathcal{A}_{U_n}}{\mathrm{ARGMIN}} \|A - X\|_F$$

- PRELIMINARY: $\|Y\|_F = \left(\sum_{j,k=1}^{n} |Y_{j,k}|^2 \right)^{1/2}$ (EASY TO COMPUTE)
  $Y \in M_n(\mathbb{C})$

- IF Q IS UNITARY ($Q^* Q = I$), THEN Q REPRESENTS AN ISOMETRY: $x \in \mathbb{C}^n$

$$\|Qx\|_2 = \|y\|_2 = \sqrt{y^* y} = \sqrt{(Qx)^* Qx} = \sqrt{x^* Q^* Q x} = \sqrt{x^* x} = \|x\|$$

G1

9-3-21

$$\|QA\|_F = \|A\|_F$$

Q unitary

$$\|AQ\|_F = \|A\|_F$$

CONSEQUENCE OF THE DEF. AND OF THE FACT THAT UNITARY MATRICES REPRESENT ISOMETRIES

$$\Updownarrow$$

$\|\cdot\|_F$ IS <u>UNITARILY INVARIANT</u>

SO NOW LET US SEE THE CONSEQUENCE OF THE FACT THAT $\|\cdot\|_F$ IS U.I. ~~TO~~ FOR OUR PROBLEM

H1

9-3-21

$P_{U_m}(A)$ WHICH IS THE MINIMIZER OF

$\| A - X \|_F$

$X \in \mathcal{A}_{U_m}$, $X = U_m D U_m^*$, D DIAGONAL

$$\| A - X \|_F = \| A - U_m D U_m^* \|_F =$$

$$= \| U_m (U_m^* A U_m - D) U_m^* \|_F =$$

$$= \| U_m^* A U_m - D \|_F$$

U_m
U_m^* ARE
UNITARY

OPTIMAL CHOICE, THAT IS THE MINIMIZER,

IS $D_{opt} = \mathrm{DIAG}(U_m^* A U_m)$

$$P_{U_m}(A) = U_m \,\mathrm{DIAG}(U_m^* A U_m)\, U_m^*$$

(I 1)

9-3-21

WHEN $U_m = F_m$ THAT IS FOR CIRCULANT MATRICES THIS "FROBENIUS OPTIMAL" PRECONDITIONER IS ALSO CALLED T. CHAN PRECONDITIONER (TOEPLITZ PRODUCTS)...

— FOR SPARSE MATRICES HUCKLE, BENZI WORKED A LOT ON THIS PRECONDITIONING.

(L1)

9-3-21

GIVEN $A \in M_n(\mathbb{C})$, GIVEN $\mathcal{A}_{U_n}$ MATRIX ALGEBRA WITH UNITARY TRANSFORM THEN FROBENIUS OPTIMAL PRECONDITIONER (OR APPROXIMATION) IS COMPUTED AS

$$P_{U_n}(A) = U_n \operatorname{diag}(U_n^* A U_n) U_n^* \quad (*)$$

$P_{U_n}: M_n(\mathbb{C}) \to \mathcal{A}_{U_n} \subseteq M_n(\mathbb{C})$ IS AN OPERATOR.

- P_{U_n} IS LINEAR: YOU SEE IT FROM $(*)$

$$P_{U_n}(\alpha A + \beta B) = \alpha P_{U_n}(A) + \beta P_{U_n}(B)$$

- P_{U_n} IS ALSO POSITIVE: IF $A = A^*$ AND IS NONNEGATIVE DEFINITE THEN $U_n^* A U_n$ IS STILL HERMITIAN AND NONNEGATIVE DEFINITE

(11) 9-3-21

THEREFORE $\mathrm{diag}(U_n^* A U_n)$ IS A NONNEGATIVE DIAGONAL MATRIX AND FINALLY

$$P_{U_n}(A) = U_n \,\mathrm{diag}(U_n^* A U_n)\, U_n^*$$

IS HERMITIAN AND NONNEGATIVE DEFINITE.

IN PARTICULAR, IF $A = T_n(f)$ THEN

$$P_{U_n}(T_n(f)) : L^1(-\pi,\pi) \longrightarrow \mathcal{A}_{U_n} \subseteq M_n(\mathbb{C})$$

AND THIS IS ANOTHER LPO.

FOR THIS SPECIFIC MATRIX-VALUED LPO THERE IS A SPECIAL KOROVKIN THEOREM.

(N1) 9-3-21

$\{X_n\}_n$ SEQUENCE OF MATRICES OF INCREASING SIZE $X_n \in M_n(\mathbb{C})$ HAS A (PROPER) STRONG CLUSTER AT α IN THE EIGENVALUE SENSE IFF $\forall_{\varepsilon>0} \exists C_\varepsilon$ SUCH THAT

$$\#\{\lambda_j(X_n) : |\lambda_j(X_n) - \alpha| > \varepsilon\} \leq C_\varepsilon$$

WHEN $\{X_n\}_n \overset{\longleftrightarrow}{}$ IS MADE BY POSITIVE DEF. MATRICES AND $\alpha = 1$, THE USE OF THE CONJ. GRADIENT ~~W~~ WILL PRODUCE ON $C_\varepsilon + 1$ ITERATIONS FOR CONVERGENCE

(AXELSSON - LINDSKÖG ESTIMATES)

01

9-3-21

Theorem (Korovkin-Toeplitz)

Let $\{A_{U_n}\}$ a sequence of matrix algebras with unitary transform: $A_{U_n} = \{X = U_n D U_n^*,\ D \text{ diagonal}\}$, $U_n^* U_n = 1$. Consider $T_n(e^{i\theta}) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}$ the transpose of basic Jordan block.

Compute the eigenvalues $\lambda_j\left(P_{U_n}(T_n(e^{i\theta}))\right) = \left(U_n^* T_n(e^{i\theta}) U_n\right)_{jj}$

If $\max_j \left| \lambda_j\left(P_{U_n}(T_n(e^{i\theta}))\right) - e^{i\frac{2\pi j}{n}} \right| \leq \frac{C}{n}$, C constant

then $\forall f \in C_{2\pi}$ = continuous and 2π-periodic

$\{T_n(f) - P_{U_n}(T_n(f))\}_n$ is strongly clustered at zero.

(P1)

9-3-21

IF IN ADDITION $f \in C_{2\pi}$ IS ALSO POSITIVE, THEN

$$\left\{ P^{-1}_{U_n} (T_n(f)) \, T_n(f) \right\}_n$$ HAS A STRONG CLUSTER AT 1. $\blacksquare$

THE POWER OF THE THEOREM IS THE ASSUMPTION WHICH IS VERY SIMPLE TO CHECK. LET US MAKE A CHECK JUST FOR CONVINCING OURSELVES.

Q1

9-3-21

$U_n = F_n$, $\mathcal{A}_{U_n}$ = CIRCULANTS

$$T_n(e^{i\theta}) = \begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \quad X \in \mathcal{A}_{U_n}$$

$$X = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_1 \\ c_1 & c_0 & \ddots & \vdots \\ \vdots & c_1 & \ddots & c_{n-1} \\ c_{n-1} & \cdots & c_1 & c_0 \end{bmatrix}$$

$$\min \| T_n(e^{i\theta}) - X \|_F$$

X_{OPT}

OPTIMAL c_0? $\quad c_0 = 0 = c_2 = c_3 = \cdots = c_{n-1}$

$c_1 = 1 - \frac{1}{n}$

$$P_{U_n}(T_n(e^{i\theta})) = \left(1-\frac{1}{n}\right) \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} = \left(1-\frac{1}{n}\right) Z_1, \quad \lambda_j(Z_1) = e^{i\frac{2\pi j}{n}}$$

$$\max_j \left| \left(1-\frac{1}{n}\right) e^{i\frac{2\pi j}{n}} - e^{i\frac{2\pi j}{n}} \right| = \max_j \left| \left(1-\frac{1}{n}\right) - 1 \right| = \frac{1}{n}$$

(R1) 9-3-21

FEW WORDS ON BAND-TOEPLITZ PRECONDITIONING
USING GLT CONCEPTS WE HAVE PROVEN

TH

IF f IS REAL VALUED A.E AND $m = \text{ess}\inf f$, $M = \text{ess}\sup f$ THEN $(m < M)$

- $\lambda_j(T_n(f)) \in (m, M)$
- $\lambda_{\min}(T_n(f)) - m \sim \frac{1}{n^\alpha}$ IF $f(\theta) - m \sim |\theta - \theta_0|^\alpha$, $\alpha > 0$

TH

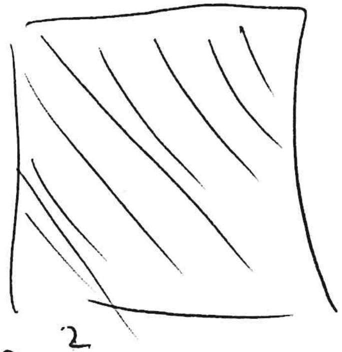
IF f IS NONNEGATIVE AND g IS NONNEGATIVE. DEFINE $r = \text{ess}\inf f/g$, $R = \text{ess}\sup f/g$ $(r < R)$

- $\lambda_j(T_n^{-1}(g)\, T_n(f)) \in (r, R)$
- $\lambda_{\min}(T_n^{-1}(g)\, T_n(f)) - r \sim \frac{1}{n^\alpha}$ IF $\frac{f}{g}(\theta) - r \sim |\theta - \theta_0|^\alpha$, $\alpha > 0$

S1

9-3-21

In this approximation of the Laplacian using the Sinc-Galerkin method (kind of spectral method), the following dense matrix appear as coefficient matrix

$$T_n = T_n(\theta^2) =$$  .

Now
$$1 \leq \frac{\theta^2}{4\sin^2\left(\frac{\theta}{2}\right)} \leq \frac{\pi^2}{4}$$
and

$$\Delta_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix} = T_n(2-2\cos\theta) = T_n\left(4\sin^2\left(\frac{\theta}{2}\right)\right)$$

T1

9-3-21

THEREFORE, BY THE THEOREM ON THE RIGHT OF R1, WE DEDUCE THAT

$$\lambda_j \left(\Delta_n^{-1} T_n \right) \in \left(1, \frac{\pi^2}{4} \right) \quad \forall n$$

$$\Downarrow$$

PCG REACHES THE SOLUTION WITHIN $O(n \log n)$ ARITHMETIC OPERATIONS