A

SERRA | 8-3-2021

- TOEPLITZ
  - VANDERMONDE
    - FOURIER
      - CIRCULANTS
        - OTHER MATRIX ALGEBRAS WITH UNITARY TRANSFORM

- LARGE MATRICES COMING FROM CONSTANT COEFF. DIFFERENTIAL OPERATORS, FRACTIONAL DIFF. OPERATORS

- LARGE MATRICES COMING FROM VARIABLE COEFF. DIFF. OPERATORS, FRACTIONAL DIFF. OPERATORS

- ONE LEVEL SETTING: 1D PROBLEMS
- MULTILEVEL SETTING: dD PROBLEMS, WHICH ARE MORE CHALLENGING AND WITH SPECIFIC DIFFICULTIES (S., TYRTYSHNIKOV, SIMAX '99)

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\# T_n IS A TOEPLITZ MATRIX IF

$(T_n)_{j,k} = a_{j-k}$ (SHIFT-INVARIANT)

$$T_n = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{1-n} \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix} \in M_n(\mathbb{C}),$$

$2n-1$ PARAMETERS $a_{1-n}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$

\# $f \in L^1(-\pi,\pi)$,

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\,d\theta$$ FOURIER COEFF.

$$T_n(f) = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{1-n} \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix}, \quad \{T_n(f)\}_n$$

TOEPLITZ MATRIX, TOEPLITZ MATRIX SEQUENCE GENERATED BY f

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FOR EXAMPLE YOU KNOW THE DISCRETE LAPLACIAN IN 1D, OBTAINED BY FINITE DIFFERENCES

$$L_{n^2} = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}, \quad f(\theta) = 2 - e^{i\theta} - e^{-i\theta} = 2 - 2\cos(\theta)$$

$$L_n = T_n(2 - 2\cos\theta)$$

$2 - 2\cos\theta$ IS REAL VALUED -

- REAL SYMMETRIC
- $2 - 2\cos\theta \geq 0$, $\max 2 - 2\cos\theta = 4$

$$0 < \lambda(L_n) < 4$$

- $2 - 2\cos\theta = 4\sin^2\left(\frac{\theta}{2}\right) \sim \theta^2$

$$\lambda_{min}(L_n) \sim \frac{1}{n^2}$$

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VANDERMONDE MATRICES

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INTERPOLATION PROBLEM:

YOU HAVE $z_0, \dots, z_m \in \mathbb{C}$, VALUES $f_0, \dots, f_m$

YOU WANT TO FIND $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$

SUCH THAT

$$p(z_j) = f_j \, , \quad j = 0, \dots, m \qquad (*)$$

NOW IF WE REACH THE EXPRESSION OF p, YOU FIND THAT $(*)$ IS EQUIVALENT TO THE FOLLOWING

$$\sum_{k=0}^{m} a_k z_j^k = f_j \qquad (**)$$

AND $(**)$ IS JUST A LINEAR SYSTEM OF $m+1$ EQUATIONS IN $m+1$ UNKNOWNS, $a_0, a_1, \dots, a_m$

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THE LINEAR SYSTEM (AND THE INTERP. PROBLEM) HAS UNIQUE SOLUTION $\Longleftrightarrow$ IFF THE COEFFICIENT MATRIX (VANDERMONDE MATRIX) IS INVERTIBLE:

$$V_m = \begin{bmatrix} 1 & z_0 & z_0^2 & \cdots & z_0^m \\ 1 & z_1 & z_1^2 & \cdots & z_1^m \\ \vdots & & & & \\ 1 & z_m & z_m^2 & \cdots & z_m^m \end{bmatrix}.$$

- PROPOSITION V_m IS INVERTIBLE IF AND ONLY IF $z_0, z_1, \dots, z_m$ ARE PAIRWISE DISTINCT

- OBSERVATION: FOR SURE IF $z_{J_1} = z_{J_2}$, $J_1 \neq J_2$ THEN V_m HAS $ROW(J_1) = ROW(J_2)$ AND $\det(V_m) = 0$

(F)

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WE PROVE A MORE PRECISE RESULT

__THEOREM__ $\det(V_m) = \prod_{\substack{k>j \\ j,k=0,\dots,m}} (z_k - z_j)$ (~)

__PROOF__ (BY INDUCTION + FUNDAMENTAL THEOREM OF ALGEBRA)

__$m=1$__ :- $V_1 \in M_2(\mathbb{C})$, $V_1 = \begin{pmatrix} 1 & z_0 \\ 1 & z_1 \end{pmatrix}$

$\det(V_1) = z_1 - z_0$. TRUE: (~) IS TRUE!

- NOW WE CONSIDER $m > 1$ AND WE USE INDUCTION THAT IS WE ASSUME

$\det V_{m-1} = \prod_{\substack{k>j \\ j,k=0,\dots,m-1}} (z_k - z_j)$

WE DEFINE $V_n : \mathbb{C} \to M_{n+1}(\mathbb{C})$

$$V_n(z) = \left[\begin{array}{cccc|c} 1 & z_0 & \cdots & & z_0^n \\ 1 & z_1 & \cdots & & z_1^n \\ \vdots & & V_{n-1} & & \\ 1 & z_{n-1} & \cdots & & z_{n-1}^n \\ \hline 1 & z & \cdots & & z^n \end{array}\right]$$

(FOR NON PAIRWISE DISTINCT NODES FORMULA ⊙ IS TRIVIAL THAT $\det V_n = 0$)

NOW WE ASSUME $z_0, z_1, \ldots, z_n$ PAIRWISE DISTINCT

1) $V_n(z_n) = V_n$

2) BY EXPANDING THE DETERMINANT $V_n(z)$ STARTING FROM POSITION $(n+1, n+1)$

$$\det V_n(z) = z^n \det(V_{n-1}) + z^{n-1} c_{n-1} + \cdots z c_1 + c_0$$

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FROM 2) WE KNOW THAT $\det V_m(z)$ IS A POLYNOMIAL OF DEGREE EQUAL TO M

$$d_m(z) = \det V_m(z) = z^m \Big(\underbrace{\prod_{\substack{k>j \\ j,k=0,\dots,m-1}} (z_k - z_j)}_{\neq 0} \Big) + c_{m-1} z^{m-1} + \dots + c_1 z + c_0$$

INDUCTIVE ASSUMPTION

Now WE KNOW ALL THE ROOTS OF $d_m(z)$:

$$d_m(z_0) = d_m(z_1) = \dots = d_m(z_{m-1}) = 0$$

FUND. TH. OF ALGEBRA

$$\Downarrow$$

$$d_m(z) = \Big(\prod_{\substack{k>j \\ j,k=0,\dots,m-1}} (z_k - z_j) \Big) (z - z_0)(z - z_1) \dots (z - z_{m-1})$$

$$\Rightarrow \det V_m = \det (V_m(z_m)) = d_m(z_m) = \prod_{\substack{k>j \\ j,k=0,\dots,m}} (z_k - z_j)$$

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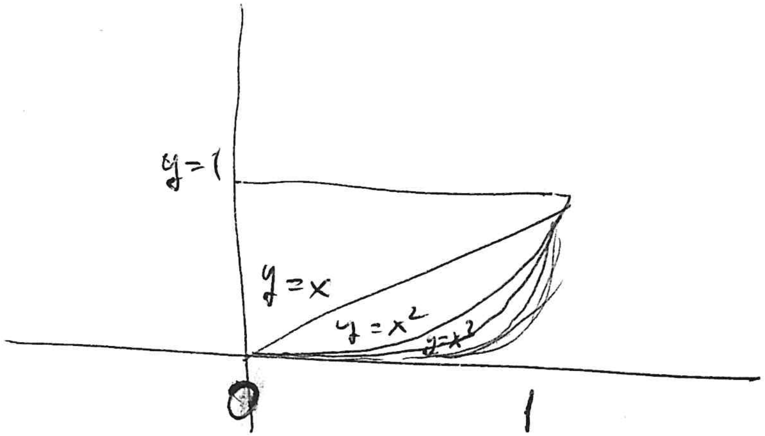
$z_0, z_1, \dots, z_m$ PAIRWISE DISTINCT $\iff$ V_m IS INVERTIBLE



$1, x, x^2, \dots, x^m$ ARE LINEARLY INDEPENDENT (CANONICAL BASIS OF $\mathbb{P}_m$), BUT THEY ARE "NUMERICALLY" DEPENDENT

THIS OBSERVATION REFLECTS IN THE FOLLOWING STATEMENT

<u>THEOREM</u>: LET $x_0, x_1, \dots, x_m \in \mathbb{R}$, CONSIDER THE RELATED VANDERMONDE MATRIX WHEN THEY ARE PAIRWISE DISTINCT: THEN $\exists\, c > 1$ <u>s.t.</u> $\mu_2(V_m) = \|V_m\| \cdot \|V_m^{-1}\| \geq c^m$ $\blacksquare$
(S., BUHI 2007)

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FOURIER MATRIX

$$F_n = \frac{1}{\sqrt{n}} \left(e^{-i \frac{2\pi JK}{n}} \right)_{J,K=0}^{n-1}$$

LET'US MAKE AN OBSERVATION:

TAKE $z_J = e^{-i\frac{2\pi J}{n}}$, $J = 0,..,n-1$. THESE ARE THE

n ROOTS OF UNITY: $z^n = 1$; ALL THE LOCATIONS ARE $z_J = e^{-i\frac{2\pi J}{n}}$, $J = 0,..,n-1$

$$F_n = \frac{1}{\sqrt{n}} V_{n-1}$$

, V_{n-1} VANDERMONDE MATRIX ~~WITH~~ WITH RESPECT TO THE NODES $z_0, z_1, .., z_{n-1}$.

(M)

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IS THIS SPECIAL VANDERMONDE MATRIX WELL-CONDITIONED? YES....

IT IS PERFECTLY CONDITIONED

$$\mu_2 (F_n) = 1$$

PHYLOSOPHIC VIEWPOINT: WHY?

BECAUSE THE ROOTS OF UNITY ~~BE~~LONG TO UNIT CIRCLE: $e^{i\theta}$

CONSEQUENTLY THE MONOMIALS $1, z, z^2, \ldots z^n$ ARE THE FUNCTIONS $1, e^{i\theta}, e^{2i\theta}, \ldots, e^{ni\theta}$ WHICH ARE PART OF AN ORTHONORMAL SYSTEM!

(N)

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PROPERTIES OF F_n

1) $F_n = F_n^T$ (COMPLEX SYMMETRIC)

2) $F_n^* F_n = I$ (F_n IS UNITARY $\Rightarrow \mu_2(F_n) = 1$)

3) $F_n^* = \overline{F_n} = F_n^{-1} = \Pi F_n = F_n \Pi$

$$\Pi = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & 1 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ 0 & 1 & & \end{array}\right]$$

OBSERVATION 1] FROM 3) IT IS EVIDENT THAT WE DO NOT NEED AN INVERSE FOURIER TRANSFORM FROM A COMPUTATIONAL VIEWPOINT

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- TAKE THE FOURIER COEFF. OF $f \in C_{2\pi}$ (CONTINUOUS AND 2π- PERIODIC) : $a_0, a_1, \ldots, a_{n-1}$

- $$a_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, e^{-ij\theta}\, d\theta$$

COMPUTE $a_0, a_1, \ldots, a_{n-1}$ USING THE COMPOSITE TRAPEZOIDAL RULE:

FIND A RELATION BETWEEN THE FORMULA THAT YOU OBTAIN AND THE MATRIX F_n.

(P)

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PROOF (OF THE THEN PROPERTIES)

1) SINCE $(F_m)_{j,k} = \frac{1}{\sqrt{m}} e^{-i\frac{2\pi jk}{m}}$ IT IS CLEAR THAT $(F_m)_{j,k} = (F_m)_{k,j}$ $\forall j,k = 0,\ldots,m-1$ THAT IS

$$F_m^T = F_m$$

2) LET US CONSIDER $F_m^* F_m$. SINCE $F_m^* = \overline{(F_m^T)}$ IT FOLLOWS THAT $F_m^* = \overline{F_m}$ (FROM POINT 1)).

THEN LET US COMPUTE

$$(\overline{F_m} F_m)_{s,t} = \begin{cases} \frac{1}{m}\sum_{k=0}^{m-1} e^{i\frac{2\pi sk}{m}} e^{-i\frac{2\pi sk}{m}} = \frac{1}{m}(1+1+\cdots 1) = 1 & s=t \\ \frac{1}{m}\sum_{k=0}^{m-1} e^{\frac{2\pi k i}{m}(s-t)} = \frac{1}{m}\sum_{k=0}^{m-1}\left[e^{i\frac{2\pi}{m}(s-t)}\right]^k & s\neq t \end{cases}$$

8) IF YOU CALL $\omega_r = e^{-i\frac{2\pi}{m}r}$ THEN

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$$\left(\overline{F_m} F_m\right)_{s,t} = \frac{1}{m} \sum_{k=0}^{m-1} \omega_{s-t}^{k} =$$

$s \neq 0$

$$= \frac{1}{m} \frac{\omega_{s-t}^{m} - 1}{\omega_{s-t} - 1}$$

BUT $\omega_{s-t} \neq 1$ BECAUSE $s \neq t$ &

$\omega_{s-t}^{m} = 1$

$$\Downarrow \quad s \neq t$$

$$\left(F_m^{*} F_m\right)_{s,t} = \left(\overline{F_m} F_m\right)_{s,t} = 0$$

IN CONCLUSION $F_m^{*} F_m = I$

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3) $$F_n = \frac{1}{\sqrt{n}} \left[\hat{v}_0 \,\middle|\, \hat{v}_1 \,\middle|\, \cdots \,\middle|\, \hat{v}_{n-1} \right] , \quad \hat{v}_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\underset{\substack{\downarrow \\ \text{USING 1)}}}{F_n^*} = \overline{F_n} = \frac{1}{\sqrt{n}} \left[\hat{v}_0 \,\middle|\, \overline{\hat{v}_1} \,\middle|\, \cdots \,\middle|\, \overline{\hat{v}_{n-1}} \right] = F_n \overbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & 1 \\ \vdots & & ⋰ & \\ 0 & 1 & & \end{bmatrix}}^{\Pi}$$

$$\overline{\hat{v}_k} = \overline{\left(e^{-i \frac{2\pi j k}{n}} \right)_{j=0}^{n-1}} = \left(e^{i \frac{2\pi j k}{n}} \right)_{j=0}^{n-1} = \left(e^{i \frac{2\pi j (k-n)}{n}} \right)_{j=0}^{n-1} =$$

$k = 1, \dots, n-1$

$$= \left(e^{-i \frac{2\pi j (n-k)}{n}} \right)_{j=0}^{n-1} = \hat{v}_{n-k}$$

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WE HAVE PROVED

$$F_m^* = F_m \Pi$$

BUT USING 1) WE HAVE

$$F_m^* = (F_m \Pi)^T = \Pi^T F_m^T$$
$$= \Pi F_m$$