A1 TOEPLITZ
VANDERMONDE
FOURIER

SVEZIA 8-3-21

COMPUTATIONAL PART: DISCRETE FOURIER TRANSFORM
FAST FOURIER TRANSFORM

DEFINITION

- $x \in \mathbb{C}^m \xmapsto[\text{DFT}]{} y = F_m x$

- $\tilde{x} \in \mathbb{C}^m \xmapsto[\text{IDFT}]{} \tilde{y} = F_m^{-1} \tilde{x} = F_m^{*} \tilde{x} = \overline{F} \tilde{x} = \Pi F_m \tilde{x} = F_m \Pi \tilde{x}$

$$F_m = \frac{1}{\sqrt{m}} \left(e^{-i \frac{2\pi jk}{m}} \right)_{j,k=0}^{m-1} = \frac{1}{\sqrt{m}} W_m$$

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THE AIM, GIVEN $x \in \mathbb{C}^n$, IS TO COMPUTE $y = W_n x$. WE DO THAT UNDER THE SPECIAL ASSUMPTION THAT $n = 2^J$, J POSITIVE INTEGER.

$n = 2^J$

$$y = W_n x = \left(\sum_{k=0}^{n-1} e^{-i \frac{2\pi j k}{n}} x_k \right)_{j=0}^{n-1}$$

OF COURSE, SINCE W_n IS DENSE, IN PRINCIPLE THE STANDARD ALGORITHM WILL REQUIRE n^2 MULTIPLICATION AND n^2 ADDITION ($\sim O(n)$ OPERATIONS)

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THE AIM IS TO REDUCE THIS QUADRATIC COST SUBSTANTIALLY

$$Y_J = \sum_{k=0}^{m-1} e^{-i\frac{2\pi JK}{m}} x_K = \sum_{k=0}^{\frac{m}{2}-1} e^{-i\frac{2\pi J(2K)}{m}} x_{2K} + \sum_{k=0}^{\frac{m}{2}-1} e^{-i\frac{2\pi J}{m}(2k+1)} x_{2k+1}$$

EVEN-ODD SEPARATION OF THE INDICES

$$= \sum_{k=0}^{\frac{m}{2}-1} e^{-i\frac{2\pi JK}{m/2}} x_{2K} + \left(\sum_{k=0}^{\frac{m}{2}-1} e^{-i\frac{2\pi JK}{m/2}} x_{2k+1} \right) e^{-i\frac{2\pi J}{m}}$$

$J = 0, \ldots, \frac{m}{2} - 1$

$$\begin{pmatrix} Y_0 \\ \vdots \\ Y_{m/2-1} \end{pmatrix} = W_{\frac{m}{2}} X_{EVEN} + D\, W_{\frac{m}{2}} X_{ODD}$$

$$D = \underset{0 \le J \le \frac{m}{2}-1}{\mathrm{DIAG}} \left(e^{-i\frac{2\pi J}{m}} \right)$$

$$X_{EVEN} = \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{m-2} \end{pmatrix}, \quad X_{ODD} = \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{m-1} \end{pmatrix}$$

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$$\begin{pmatrix} x_0 \\ \vdots \\ x_{\frac{m}{2}-1} \end{pmatrix} = W_{\frac{m}{2}} x_{even} + D\, W_{\frac{m}{2}} x_{odd}$$

$$J = \hat{J} + \frac{m}{2} \quad , \quad \hat{J} = 0, \to \frac{m}{2} - 1$$

$$Y_J = \sum_{k=0}^{\frac{m}{2}-1} e^{-i\frac{2\pi\left(\hat{J}+\frac{m}{2}\right)k}{m/2}} x_{2k} + \sum_{k=0}^{\frac{m}{2}-1} e^{-i\frac{2\pi\left(\hat{J}+\frac{m}{2}\right)(2k+1)}{m}} x_{2k+1}$$

$$= \sum_{k=0}^{\frac{m}{2}-1} e^{-i\frac{2\pi \hat{J} k}{m/2}} \underbrace{e^{-i 2\pi k}}_{1} x_{2k} + \left(\sum_{k=0}^{\frac{m}{2}-1} e^{-i\frac{2\pi\left(\hat{J}+\frac{m}{2}\right)2k}{m}} x_{2k+1} \right) e^{-i\frac{2\pi(\hat{J}+}{m}}$$

$$= \sum_{k=0}^{\frac{m}{2}-1} e^{-i\frac{2\pi \hat{J} k}{m/2}} x_{2k} - e^{-i\frac{2\pi \hat{J}}{m}} \sum_{k=0}^{\frac{m}{2}-1} e^{-i\frac{2\pi \hat{J} k}{m/2}} x_{2k+1}$$

$$\begin{pmatrix} Y_{\frac{m}{2}} \\ \vdots \\ Y_{m-1} \end{pmatrix} = W_{\frac{m}{2}} x_{even} - D\, W_{\frac{m}{2}} x_{odd}$$

$$e^{-i\frac{2\pi\hat{J}}{m}} \cdot e^{-i\frac{\cancel{2}\pi \cancel{m}}{\cancel{m}\cancel{2}}} = $$

$$= -e^{-i\frac{2\pi\hat{J}}{m}}$$

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$$y = W_n x = \begin{pmatrix} W_{\frac{n}{2}} & D W_{\frac{n}{2}} \\ W_{\frac{n}{2}} & -D W_{\frac{n}{2}} \end{pmatrix} \begin{pmatrix} x_{even} \\ x_{odd} \end{pmatrix} =$$

$$= \begin{pmatrix} W_{\frac{n}{2}} & D W_{\frac{n}{2}} \\ W_{\frac{n}{2}} & -D W_{\frac{n}{2}} \end{pmatrix} \Pi_{E-O}\, x \quad , \forall x \in \mathbb{C}^n$$

$$W_n = \begin{pmatrix} W_{\frac{n}{2}} & D W_{\frac{n}{2}} \\ W_{\frac{n}{2}} & -D W_{\frac{n}{2}} \end{pmatrix} \Pi_{E-O} \quad , \quad \Pi_{EO} = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \hline 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ & \cdots & & & & 1 \end{pmatrix}$$

THIS PERMUTATION IS RELATED TO "CUTTING" MATRICES IN MULTIGRID OR DECIMATION MATRICES IN WAVELETS...

(F1) FOR COMPUTING $W_n x$ 8-3-21

YOU COMPUTE $z = W_{\frac{n}{2}} x_{even}$, $t = W_{\frac{n}{2}} x_{odd}$,

$\tau = D t$;

$$y = \begin{pmatrix} z + \tau \\ z - \tau \end{pmatrix};$$

COMPUTATIONAL COST? ~~H~~HOW MANY OPERATIONS?

$M(n)$ MULTIPLICATIVE COST, $M(2) = 0$

$A(n)$ ADDITIVE COST, $A(2) = 2$

$$W_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad n = 2^{\ell}$$

$$M(n) = 2\,M\left(\frac{n}{2}\right) + \frac{n}{2} = 2\left[2M\left(\frac{n}{4}\right) + \frac{n}{4}\right] + \frac{n}{2} =$$

$$2\left[2\left[2M\left(\frac{n}{8}\right) + \frac{n}{8}\right]\right] + \frac{n}{2} + \frac{n}{2} = \cdots = 2^{i} M\left(\frac{n}{2^i}\right) + i\,\frac{n}{2}$$

i-th LEVEL

LAST $i = \ell - 1 = \log_2(n) - 1$

$$= 2^{\ell-1} M(2) + (\ell-1)\frac{n}{2} = \frac{n}{2}(\log_2(n$$

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$$A(n) = 2A\left(\frac{n}{2}\right) + n = \cdots =$$

$$= n \log_2(n)$$

TOTAL ARITHMETIC COST IS

$$\boxed{\frac{3}{2} n \log_2 n - \frac{n}{2}} \quad , \quad n = 2^d$$

— THE SAME COST FOR THE INVERSE FOURIER TRANSFORM

STEP1 : CIRCULANTS $\longleftrightarrow$ AND FOURIER

STEP2 : TOEPLITZ VS CIRCULANTS

STEP3 : A FAST FOURIER ALGORITHM USING A FAST TOEPLITZ ALGORITHM | NO RESTRICTIONS ON THE SIZE N

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- RELATIONSHIPS BETWEEN F_m AND THE CIRCULANTS
- CIRCULANTS FORM AN ALGEBRA OF MATRICES THAT IS A SET CLOSE UNDER LINEAR COMBINATIONS, PRODUCT AND INVERSE (WHENEVER THE INVERSE EXISTS)

LET US WRITE A GENERIC CIRCULANT MATRIX

$$A_m = \left(a_{(j-k) \bmod m} \right)_{j,k=0}^{m-1} = \begin{pmatrix} a_0 & a_{m-1} & \cdots & a_2 & a_1 \\ a_1 & a_0 & & & a_2 \\ a_2 & a_1 & \ddots & & \vdots \\ \vdots & a_2 & \ddots & \ddots & a_{m-1} \\ a_{m-1} & \cdots & a_2 & a_1 & a_0 \end{pmatrix} =$$

↘ WE SEE IMMEDIATELY THE STRUCTURE OF VECTOR SPACE

$$= a_0 I + a_1 Z_1 + a_2 Z_2 + \cdots + a_{m-1} Z_{m-1}$$

$$Z_1 = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0 & & 1 & 0 \\ 0 & & & 1 \\ 1 & \ddots & & \\ 0 & & 1 \; 0 & 0 \end{pmatrix}, \ \ldots, \ Z_{m-1} = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{pmatrix} = Z_1^T$$

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THE DIMENSION OF THE VECTOR SPACE OF CIRCULANTS IS EXACTLY n SINCE, $I, Z_1, Z_2, \dots, Z_{n-1}$ ARE LINEARLY INDEPENDENT (Z_K HAS 'ONES' WHEN ALL THE OTHER HAVE 'ZEROS')

IF WE LOOK A Z_1 AS A "GRAPH" MATRIX THE Z_1 REPRESENTS A "GAME"

NOW IF WE APPLY THIS "GAME" TWO TIMES, THEN WE SEE THAT $Z_1^2 = Z_2$, $Z_1^3 = Z_3$, $Z_1^4 = Z_4$, $\dots$, $Z_1^{n-1} = Z_{n-1}$

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THEREFORE

$$A_m = a_0 I + a_1 Z_1 + a_2 Z_1^2 + \cdots + a_{m-1} Z_1^{m-1}$$

BUT Z_1 IS A PERMUTATION MATRIX AND THEREFORE ITS TRANSPOSE IS ITS INVERSE : $Z_1^{m-1} = Z_1^{-1} \Longrightarrow$

$$\underline{\underline{Z_1^{K} = Z_1^{K \bmod m}}}$$

AS A CONSEQUENCE IF A_m AND B_m ARE CIRCULANTS THAT IS $A_m = \sum_{j=0}^{m-1} a_j Z_1^j$, $B_m = \sum_{j=0}^{m-1} b_j Z_1^j$

THEN

$$X = A_m \cdot B_m = \sum_{j=0}^{2m-2} c_j Z_1^j, \quad c_j = \sum_{s+t=j} a_s \cdot a_t$$

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IN CONCLUSION

$$X = A_m B_m = \sum_{J=0}^{2m-2} c_J Z_1^{J \bmod m} \text{ IS CIRCULANT}$$

THE SPACE IS CLOSE UNDER MULTIPLICATION

WHAT ABOUT INVERSION?

A_m IS CIRCULANT, $\det(A_m) \neq 0$; IS IT TRUE THAT A_m^{-1} IS CIRCULANT?

- GIVEN THE CAYLEY-HAMILTON THEOREM, THE PROOF IS ELEMENTARY

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THEOREM (CAYLEY-HAMILTON)

$X \in M_n(\mathbb{C})$, $p_X(\lambda) = \det(X - \lambda I)$ CHARACTERISTIC POLYNOMIAL. THEN $p_X(\lambda) = (-1)^n \lambda^n + \dots + \det(X)$

AND

$$p_X(X) = O_{M_n(\mathbb{C})}$$

<u>COROLLARY</u>: IF X IS INVERTIBLE THEN X^{-1} IS A POLYNOMIAL OF X:

BY THE CAYLEY-HAMILTON THEOREM

$$(-1)^n X^n + \dots + \det(X) \cdot I = O_{M_n(\mathbb{C})}$$

$$X^{-1} \cdot \det(X) = (-1)^{n+1} X^{n-1} + c_{n-2} X^{n-2} + \dots + c_1 X + c_0 I$$

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By combining the fact that circulants are close under linear combinations and products, since the inverse of an invertible matrix is a polynomial of the matrix itself, it follows that circulants are close also under inversion

↓

Circulants form an algebra of matrices

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SINCE $Z_1 = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$ IS SO IMPORTANT FOR CIRCULANTS (IN FACT Z_1 IS CALLED "GENERATOR" OF THE CIRCULANT ALGEBRA), LET US SEE WHAT HAPPENS TO THE PRODUCT

$$Z_1 F_n = Z_1 \cdot \frac{1}{\sqrt{n}} \left(e^{-i \frac{2\pi jk}{n}} \right)_{j,k=0}^{n-1}$$

$$Z_1 \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} v_{n-1} \\ v_0 \\ v_1 \\ \vdots \\ v_{n-2} \end{pmatrix} = \begin{pmatrix} v_{(-1) \bmod n} \\ v_{0 \bmod n} \\ \vdots \\ v_{n-2 \bmod n} \end{pmatrix}$$

$$Z_1 \left(e^{-i \frac{2\pi jk}{n}} \right)_{j=0}^{n-1} = \left(e^{-i \frac{2\pi (j-1)k}{n}} \right)_{j=0}^{n-1} = e^{i \frac{2\pi k}{n}} \left(e^{-i \frac{2\pi jk}{n}} \right)_{j=0}^{n-1}$$

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$$F_n = \left[f_0 \middle| f_1 \middle| \cdots \middle| f_{n-1} \right]$$

$$Z_1 f_k = e^{\hat{i}\frac{2\pi k}{n}} f_k$$

↳ EIGENVECTOR → EIGENVALUE

$$Z_1 F_n = F_n \operatorname*{DIAG}_{0 \le k \le n-1} \left(e^{\hat{i}\frac{2\pi k}{n}} \right)$$

$$Z_1 = F_n \operatorname*{DIAG}_{0 \le k \le n-1} \left(e^{\hat{i}\frac{2\pi k}{n}} \right) F_n^*$$

(JORDAN, SCHUR ROOT OF Z_1)

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TAKE A_n CIRCULANT

THAN $A_n = \sum_{j=0}^{n-1} a_j Z_1^j =$

$$= \sum_{j=0}^{n-1} a_j \left(F_n \, \underset{k}{\mathrm{DIAG}}\left(e^{i\frac{2\pi k}{n}}\right) F_n^* \right)^j$$

$$= \sum_{j=0}^{n-1} a_j \, F_n \left(\underset{k}{\mathrm{DIAG}}\left(e^{i\frac{2\pi k}{n}}\right)\right)^j F_n^*$$

$$= F_n \sum_{j=0}^{n-1} a_j \left(\underset{k}{\mathrm{DIAG}}\left(e^{i\frac{2\pi k}{n}}\right)\right)^j F_n^*$$

$$= F_n \left[\mathrm{DIAG}\left(\sqrt{n}\, F_n^* \underline{a}\right)\right] F_n^* \, , \quad \underline{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

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So any circulant matrix whose first column is $\underline{a} = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}$ has eigenvalues given by

$$\sqrt{n}\, F_n^{*}\, \underline{a} = \sqrt{n}\, \Pi F_n\, \underline{a} = \sqrt{n}\, F_n \Pi\, \underline{a} .$$

- So the computation of the eigenvalues costs $\frac{3}{2} n \log_2 n - \frac{n}{2}$ if n is a power of n

- Given x you want to compute $A_n x$ then you have 3 FFT w n is a power of 2 so the cost $\frac{9}{2} n \log_2 n - \frac{3}{2} n$

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IF YOU WANT TO SOLVE A LINEAR SYSTEM $A_m x = b$ WITH A_m CIRCULANT THEN

STEP 1: COMPUTE THE EIG. OF A_m AS $\hat{a} = \sqrt{m}\, \sqrt{F_m}\, \underline{e}$. IF THERE ARE NO ZERO EIGS THEN CONTINUE

STEP 2: $F_m \,\mathrm{DIAG}(\hat{a})\, F_m^{*} x = b$

$$x = F_m \,\mathrm{DIAG}^{-1}(\hat{a})\, F_m^{*} b$$

(3 FFT ARE SUFFICIENT)

$\frac{9}{2} m \log_2 m$ IF m IS A POWER OF 2