Hands on by Stefano Serra Capizzano (March 9th, 2021)

Hands on A

Consider the matrix of size $n \geq 1$

$$A_n(a) = \begin{bmatrix} a_{\frac{1}{2}} + a_{\frac{3}{2}} & -a_{\frac{3}{2}} & & & \\ -a_{\frac{3}{2}} & a_{\frac{3}{2}} + a_{\frac{5}{2}} & -a_{\frac{5}{2}} & & \\ & \ddots & \ddots & \ddots & \\ & & & -a_{n-\frac{1}{2}} & a_{n-\frac{1}{2}} + a_{n+\frac{1}{2}} \end{bmatrix}, \quad a_t = a\left(\frac{t}{n+1}\right), \qquad (1)$$

where $a : [0,1] \to \mathbf{R}$ is a positive function (choose now a positive function).

1. Write a Matlab procedure for building the matrix $= A_n(a)$ with fixed function a (the one you have chosen) and $n \geq 1$ as a parameter.

2. Apply CG to a system $A_n(a)x = b$ (choose your data = vector b) for various n ($n = 16, 32, 64, 128, \ldots$) and fixed tolerance $\epsilon = 10^{-6}$:
   • What do you see concerning the number of required iterations for convergence?

3. Compute $\lambda_{\min}(A_n(a))$ for various n ($n = 16, 32, 64, 128, \ldots$):
   • What do you see?
   Given the fact that $\lambda_{\min}(A_n(a)) \approx \alpha(a)/n^s$, $\alpha(a), s$ positive numbers, "invent" a numerical procedure for computing s and then $\alpha(a)$.

4. Compare and discuss the results in Item 2 and the value s that you numerically detected at Item 3.

5. Make all the previous steps for the new matrix $B_n(a) = A_n(a) + 17e_ne_n^T$, where e_n is the n-th vector of the canonical basis:
   • Comment on the related results from the theoretical viewpoint.

Hands on B

Consider the same matrix $A_n(a)$ of size $n \geq 1$ as defined in (1).

1. Consider the matrix $P_n = A_n(1)$ and prove that it coincides with $T_n(2 - 2\cos(s))$ (i.e. the n-by-n Toeplitz matrix generated by the cosine polynomial $2 - 2\cos(s)$).

2. Apply CG with preconditioner P_n to a system $= A_n(a)x = b$ (choose your data vector b) for various n ($n = 16, 32, 64, 128, \ldots$) and fixed tolerance $\epsilon = 10^{-6}$:
   • What do you see concerning the number of required iterations for convergence?

3. Compute $\lambda_{\min}(P_n)$ for various n ($n = 16, 32, 64, 128, \ldots$):
   • What do you see?
   Given the fact that $\lambda_{\min}(P_n) \approx \alpha/n^s$, α, s positive numbers, "invent" a numerical procedure for computing s and then α.

4. Compute $\lambda_{\min}(P_n^{-1}A_n(a))$, $\lambda_{\max}(P_n^{-1}A_n(a))$ for various n ($n = 16, 32, 64, 128, \ldots$):
   • Which is the relation that you see between the computed values $\lambda_{\min}(P_n^{-1}A_n(a))$ and $\min\limits_{x\in[0,1]} a(x)$?
   • Which is the relation that you see between the computed values $\lambda_{\max}(P_n^{-1}A_n(a))$ and $\max\limits_{x\in[0,1]} a(x)$?

5. Compare and discuss the results in Item 2 and in Item 4.

Hands on C

Consider the same matrix $A_n(a)$ of size $n \geq 1$ as defined in (1).

1. It can be proved that the eigenvalues of $A_n^{-1}(b)A_n(a)$, for positive functions a and b, "behave" as a sampling of a/b over a uniform gridding of the domain $[0, 1]$. Write a numerical procedure for detecting numerically this phenomenon. Consider, as in the previous exercise, the case of your function a and $b \equiv 1$.

2. It can be proved that the eigenvalues of $A_n(a)$, for a positive function a, "behave" as a sampling of $a(x)(2 - 2\cos(y))$ over a uniform gridding of the domain $[0, 1] \times [0, \pi]$. Write a numerical procedure for detecting numerically this phenomenon. Consider, as in the previous exercise, the case of your function a.

Hands on D

Consider the matrix of size $n \geq 1$ whose j-th row is defined as

$$(0, \ldots, 0, a_{j-1}, -2(a_{j-1} + a_j), a_{j-1} + 4a_j + a_{j+1}, -2(a_j + a_{j+1} =), a_{j+1}, 0, \ldots, 0), \qquad (2)$$

with (j, j) position given by $a_{j-1} + 4a_j + a_{j+1}$ and where $a_t = a\left(\frac{t}{n+1}\right)$, $a : [0, 1] \to \mathbf{R}$ as in Hands on A.

1. Write a Matlab procedure for building the matrix $= A_n(a)$ with fixed function a (the one you have chosen) and $n \geq 1$ as a parameter.

2. Apply CG to a system $A_n(a)x = b$ (choose your data = vector b) for various n ($n = 16, 32, 64, 128, \ldots$) and fixed tolerance $\epsilon = 10^{-6}$:
   • What do you see concerning the number of required iterations for convergence?

3. Compute $\lambda_{\min}(A_n(a))$ for various n ($n = 16, 32, 64, 128, \ldots$):
   • What do you see?
   Given the fact that $\lambda_{\min}(A_n(a)) \approx \alpha(a)/n^s$, $\alpha(a), s$ positive numbers, "invent" a numerical procedure for computing s and then $\alpha(a)$.

4. Compare and discuss the results in Item 2 and the value s that you numerically detected at Item 3.

Hands on E

Consider the same matrix $A_n(a)$ of size $n \geq 1$ as defined in (2).

1. Consider the matrix $P_n = A_n(1)$ and prove that it coincides with $T_n((2 - 2\cos(s))^2)$ (i.e. the n-by-n Toeplitz matrix generated by the cosine polynomial $(2 - 2\cos(s))^2$).

2. Apply CG with preconditioner P_n to a system $A_n(a)x = b$ (choose your data vector b) for various n ($n = 16, 32, 64, 128, \ldots$) and fixed tolerance $\epsilon = 10^{-6}$:
   • What do you see concerning the number of required iterations for convergence?

3. Compute $\lambda_{\min}(P_n)$ for various n ($n = 16, 32, 64, 128, \ldots$):
   • What do you see?
   Given the fact that $\lambda_{\min}(P_n) \approx \alpha/n^s$, α, s positive numbers, "invent" a numerical procedure for computing s and then α.

4. Compute $\lambda_{\min}(P_n^{-1}A_n(a))$, $\lambda_{\max}(P_n^{-1}A_n(a))$ for various n ($n = 16, 32, 64, 128, \ldots$):
   • Which is the relation that you see between the computed values $\lambda_{\min}(P_n^{-1}A_n(a))$ and $\min_{x\in[0,1]} a(x)$?
   • Which is the relation that you see between the computed values $\lambda_{\max}(P_n^{-1}A_n(a))$ and $\max_{x\in[0,1]} a(x)$?

5. Compare and discuss the results in Item 2 and in Item 4.

Hands on F

Consider the dense symmetric Toeplitz matrix $T_n = T_n(f(s))$, $f(s) = s^2$, of size $n \geq 2$ whose 1-st row is given by

$$(\pi^2/3, -2, 2/2^2, -2/3^2, 2/4^2, \ldots, -2(-1)^n/(n-1)^2). \tag{3}$$

1. Write a Matlab procedure for building the matrix $= T_n$ with and $n \geq 2$ as a parameter.

2. Apply CG to a system $T_n x = b$ (choose your data vector b) for various n ($n = 16, 32, 64, 128, \ldots$) and fixed tolerance $\epsilon = 10^{-6}$:
   • What do you see concerning the number of required iterations for convergence?

3. Compute $\lambda_{\min}(T_n)$ for various n ($n = 16, 32, 64, 128, \ldots$):
   • What do you see?
   Given the fact that $\lambda_{\min}(T_n) \approx \alpha(a)/n^s$, $\alpha(a), s$ positive numbers, "invent" a numerical procedure for computing s and then $\alpha(a)$.

4. Compare and discuss the results in Item 2 and the value s that you numerically detected at Item 3.

Hands on G

Consider the same matrix T_n of size $n \geq 2$ as defined in (3).

1. Consider the matrix $P_n = T_n(2 - 2\cos(s))$ (i.e. the n-by-n Toeplitz matrix generated by the cosine polynomial $2 - 2\cos(s)$).

2. Apply CG with preconditioner P_n to a system $= T_n x = b$ (choose your data vector b) for various n ($n = 16, 32, 64, 128, \ldots$) and fixed tolerance $\epsilon = 10^{-6}$:
   - What do you see concerning the number of required iterations for convergence?

3. Compute $\lambda_{\min}(P_n^{-1}T_n)$, $\lambda_{\max}(P_n^{-1}T_n)$ for various n ($n = 16, 32, 64, 128, \ldots$):
   - Which is the relation that you see between the computed values $\lambda_{\min}(P_n^{-1}T_n)$ and $\min\limits_{s\in[0,\pi]} f(s)/p(s)$?
   - Which is the relation that you see between the computed values $\lambda_{\max}(P_n^{-1}T_n)$ and $\max\limits_{s\in[0,\pi]} f(s)/p(s))$?

4. Which relation do you see between the results in Item 2 and those in Item 3?