SVD: Estimates of singular values

Sources:
C.R. Johnson
A Gershgorin-type lower bound for the smallest singular value
Linear Algebra Appl., 112 (1989), pp. 1-37

J.M. Varah
A lower bound for the smallest singular value of a matrix
Linear Algebra Appl., 11 (1975), pp. 3-5

L. Qi
Some simple estimates for the singular values of a matrix
Linear Algebra Appl., 56 (1984), pp. 105-119

SVD: Estimates of singular values (Varah)

$A(n,n)$ diagonally dominant both by rows and columns, $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n \geq 0$
Recall $\|A\|_2^2 \leq \|A\|_1 \|A\|_\infty$,
$\|A\|_1$, $\|A\|_\infty$ - max abs column/row sum (computable).
Let

$$\alpha = \min_k \left(|a_{kk}| - \sum_{j \neq k} |a_{kj}| \right)$$

$$\beta = \min_k \left(|a_{kk}| - \sum_{j \neq k} |a_{jk}| \right)$$

Then

$$\sigma_n \geq \sqrt{\alpha\beta}$$

$A = \begin{bmatrix} 4 & 1 \\ 3 & 5 \end{bmatrix}$, eigs=$[2.697, 6.3028]$, svd=$6.6713, 2.5482$, $\alpha = 2$, $\beta = 1$, $\sqrt{\alpha\beta} =$ 1.4142

SVD: Estimates of singular values (Liqun Qi)

$A(m,n)$, $m \geq n$

$$r_i = \sum_{j=1}^{n} |a_{ij}|,\ c_i = \sum_{j=1}^{n} |a_{ji}|,\ s_i = \max(r_i, c_i),\ a_i = |a_{ii}|\ i = 1, \cdots, \min(m,n), j \neq i$$

With the above notation, each singular value lies in certain interval.

$$\sigma_1 \geq \max(\max_{1 \leq i \leq m} \{\|a_{i,.}\|\}, \max_{1 \leq j \leq n} \{\|a_{.,j}\|\}$$

$$\sigma_m \leq \min(\min_{1 \leq i \leq m} \{\|a_{i,.}\|\}, \min_{1 \leq j \leq n} \{\|a_{.,j}\|\}$$

$A = \begin{bmatrix} 4 & 1 \\ 3 & 5 \end{bmatrix}$, svd=$6.6713, 2.5482$, [norm(A(:,1)) norm(A(:,2));norm(A(1,:)) norm(A(2,:))] 5.000000000000000 5.099019513592785 4.123105625617661 5.830951894845300

$2.5482 \leq ...,\ 2.8284 \leq 6.6713$

SVD: Demo 2: Estimates of singular values

We consider a family of bidiagonal matrices, depending on two parameters,

$$A = \begin{bmatrix} 1-\eta & \beta(1-\eta) & & & \\ & 1-\eta & \beta(1-\eta) & & \\ & & \ddots & \vdots & \\ & & & & \beta(1-\eta) \\ & & \cdots & \cdots & 1-\eta \end{bmatrix}$$

where $\eta \ll 1$. In a paper by Demmel and Kahan, 1990, it was shown that the smalles singular value of $A(\eta)$ is approximately $\beta^{1-n}(1-(2n-1)\eta)$.
We test with
$\beta = [5, 100, 10000, 10^6]$
$\eta = [0.5, 10^{-2}, 10^{-6}, 10^{-6}]$

$n = 20$