Assignments by Stefano Serra Capizzano (March 9th 2021)

Consider the matrix of size $n \geq 1$

$$A_n(a) = \begin{bmatrix} a_{\frac{1}{2}} + a_{\frac{3}{2}} & -a_{\frac{3}{2}} & & \\ -a_{\frac{3}{2}} & a_{\frac{3}{2}} + a_{\frac{5}{2}} & -a_{\frac{5}{2}} & \\ & \ddots & \ddots & \ddots \\ & & -a_{n-\frac{1}{2}} & a_{n-\frac{1}{2}} + a_{n+\frac{1}{2}} \end{bmatrix}, \quad a_t = a\left(\frac{t}{n+1}\right) \tag{1}$$

where $a : [0,1] \to \mathbf{R}$ is a positive function.

1. Prove that the matrix in (1) is the discretization of the boundary value problem

$$\begin{cases} -\left(a(x)u_x\right)_x = f\left(x\right) & \text{on } \Omega = (0,1), \\ \text{Dirichlet B.C. on } \partial\, \Omega, \end{cases} \tag{2}$$

   by centered Finite Differences of precision order 2 and step-size $h = (n+1)^{-1}$.

2. Prove that $A_n(a)$ as in (1) is positive definite.

3. Prove that $P_n^{-1}A_n(a)$ is similar to a positive definite matrix with $P_n = A_n(1)$.

4. Prove that any eigenvalue of $P_n^{-1}A_n(a)$ belongs = to $[a_*, a^*]$ with $a_* = \min_{x\in[0,1]} a(x)$ and $a^* = \max_{x\in[0,1]} a(x)$.

5. Prove the very same statements as in Items 2–4 for the matrices $A_n(a)$ and P_n as in Hands on D and E.

6. Prove that $T_n(f(s))$ with $f(s) = s^2$ and $n \geq 2$ = is the symmetric Toeplitz matrix whose central row is the one appearing in (3), Hands on F.

7. Give localization results (uniformly with respect to = the size n) for the eigenvalues of the matrices appearing in Hands on F and G.