Uppsala University
Graduate School in Mathematics and Computing
Institute for Information Technology

Numerical Linear Algebra FMB and MN1
Fall 2007

Mandatory Assignment 3.B

OBS! To the attention of the graduate students. On your choice, solve two problems from the list as follows:
– one problem among Exercises 1–3 and one problem among Exercises 4-6, or
– two problems among Exercises 4-6.

<u>Exercise</u> **1 (Comparison matrices)**
Some theory:

Definition 1 A real square nonsingular matrix $A = [a_{ij}]$ is called an M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and if A is monotone, i.e., if $A^{-1} \geq 0$ (pointwise).
For any real or complex square matrix A we denote by $\mathcal{M}(A) = [m_{ij}]$ the real matrix, called the *comparison matrix*, defined by

$$m_{ij}(A) = \begin{cases} |a_{ij}|, & i = j \\ -|a_{ij}|, & i \neq j \end{cases}$$

A matrix A, for which $\mathcal{M}(A)$ is an M-matrix, is called an H-matrix.

Theorem 1 (Ostrowski's theorem) *Let A be an H-matrix and let $\mathcal{M}(A)$ be its comparison matrix. Then $|A^{-1}| \leq \mathcal{M}(A)^{-1}$.*

The tasks:

(a) Show that every nonsingular triangular matrix is an H-matrix.

(b) Let A be nonsingular with a triangular decomposition $A = LU$. Show that

$$|A^{-1}| \leq \mathcal{M}(U)^{-1}\mathcal{M}(L)^{-1}.$$

Here $|A|$ denotes the matrix with entries $|a_{ij}|$.

Exercise 2

(a) Let $U = [u_{ij}]$ be a nonsingular upper-triangular matrix. Show that with respect to the infinity norm there holds

$$\varkappa(U) \geq \frac{\max(u_{ii})}{\min(u_{ii})}.$$

(b) Let $A = LDL^T$ be a symmetric positive definite matrix and let $D = diag(d_{ii})$. Show that with respect to the 2-norm there holds

$$\varkappa(A) \geq \frac{\max(d_{ii})}{\min(d_{ii})}.$$

Construct an example of an ill-conditioned nondiagonal symmetric positive definite matrix.

Exercise 3 The coefficient matrix of a system of linear equations $A\mathbf{x} = \mathbf{b}$ has the form $A = I - L - U$, where L and U are strictly lower and upper triangular matrices. Consider the iteration

$$\mathbf{x}^{(k+1)} = \widetilde{\mathbf{b}} + \mathbf{x}^{(k)} - (I-U)^{-1}(I-L)^{-1}A\mathbf{x}^{(k)},$$

where $\widetilde{\mathbf{b}} = (I-U)^{-1}(I-L)^{-1}\mathbf{b}$.

(a) What is the iteration matrix of the above recurrence?

(b) Show that if the method converges, it converges to the correct solution.

(c) Explain how this iterative process can be performed without explicitly forming the inverses of $I - L$ and $I - U$.

Exercise 4 Consider a matrix of size $n \geq 1$

$$A_n(a) = \begin{bmatrix} a_{\frac{1}{2}} + a_{\frac{3}{2}} & -a_{\frac{3}{2}} & & \\ -a_{\frac{3}{2}} & a_{\frac{3}{2}} + a_{\frac{5}{2}} & -a_{\frac{5}{2}} & \\ & \ddots & \ddots & \ddots \\ & & -a_{n-\frac{1}{2}} & a_{n-\frac{1}{2}} + a_{n+\frac{1}{2}} \end{bmatrix}, \quad a_t = a\left(\frac{t}{n+1}\right), \tag{1}$$

where $a : [0,1] \to \mathbf{R}$ is a positive function.

1. (**Not compulsory**) Prove that the matrix in (1) is the discretization of the boundary value problem

$$\begin{cases} -\left(a(x)u_x\right)_x = f(x) & \text{on } \Omega = (0,1), \\ \text{Dirichlet B.C. on } \partial\,\Omega, \end{cases} \tag{2}$$

by central Finite Differences of second order of accuracy and a discretization step $h = (n+1)^{-1}$.

2. Prove that $A_n(a)$ as in (1) is positive definite.

3. Prove that $P_n^{-1}A_n(a)$ is similar to a symmetric positive definite matrix with $P_n = A_n(1)$.

4. Prove that any eigenvalue of $P_n^{-1}A_n(a)$ belongs to the interval $[a_*, a^*]$ with $a_* = \min\limits_{x\in[0,1]} a(x)$ and $a^* = \max\limits_{x\in[0,1]} a(x)$.

<u>Exercise</u> 5 Consider the matrix $A_n(a)$ of size $n \geq 1$ whose j-th row is defined as

$$(0, \ldots, 0, a_{j-1}, -2(a_{j-1} + a_j), a_{j-1} + 4a_j + a_{j+1}, -2(a_j + a_{j+1} =), a_{j+1}, 0, \ldots, 0), \qquad (3)$$

with (j, j) position given by $a_{j-1} + 4a_j + a_{j+1}$ and where $a_t = a\left(\frac{t}{n+1}\right)$, $a : [0, 1] \to \mathbf{R}$ as in Exercise 4. Consider also the matrices $P_n = A_n(1)$.
Prove the very same statements as in Items 2–4 of **Exercise 4** for the matrices $A_n(a)$ and P_n.

<u>Exercise</u> 6 Consider the dense symmetric Toeplitz matrix T_n of size $n \geq 2$ whose 1-st row is given by

$$(\pi^2/3, -2, 2/2^2, -2/3^2, 2/4^2, \ldots, -2(-1)^n/(n-1)^2). \qquad (4)$$

1. Prove that $T_n = T_n(f(s))$ with $f(s) = s^2$.

2. Give localization results (uniformly with respect to the size n) for the eigenvalues of the matrices T_n.

Deadline: The solutions should be delivered to Maya Neytcheva no later than by **December 10, 2007**.
Success!

Maya (`Maya.Neytcheva@it.uu.se`, room 2307)

Any comments on the assignment will be highly appreciated and will be considered for further improvements. Thank you!