Uppsala University
Graduate School in Mathematics and Computing
Institute for Information Technology

Numerical Linear Algebra FMB and MN1
Fall 2007

Mandatory Assignment 2:

Guidelines how to prepare the assignment:
The assignment should contain six exercises from the list below, chosen as follows.

(i) Exercises 1 to 3 are mandatory for all course participants.

(i) Undergraduate students have to choose 2 among exercises 4 to 7 and one among exercises 8 to 11.

(iii) Graduate students have to choose one among exercises 4 to 7 and two among exercises 8 to 11.

Exercise 1 (Aasen's algorithm) A method to reduce a symmetric matrix to a tridiagonal one. Let $A(n \times n)$ be symmetric and partitioned in a block two-by-two form as follows:

$$A^{(1)} = A = \begin{bmatrix} a_{11} & a_{12} & \mathbf{u}^T \\ a_{21} & & \\ \mathbf{u} & & A_{22} \end{bmatrix}.$$

Assume that $a_{12} = a_{21} \neq 0$. Here A_{22} is of order $n-1$. Let

$$L_1 = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ \mathbf{0} & \ell_1 & I \end{bmatrix},$$

where $\ell_1 = -\frac{1}{a21}\mathbf{u}$. Let also $\mathbf{v}^{(1)} = \begin{bmatrix} a_{21} \\ \mathbf{u} \end{bmatrix}$.

(a) Consider the congruence transformation $A^{(2)} = L_1 A^{(1)} L_1^T$ and show that

$$A^{(2)} = L_1 A^{(1)} L_1^T = \begin{bmatrix} a_{11} & a_{12} & \mathbf{0}^T \\ a_{21} & & \\ \mathbf{0} & & A_{22}^{(2)} \end{bmatrix}$$

where $A_{22}^{(2)}$ is of order $n-1$ and

$$A_{22}^{(2)} = \begin{bmatrix} 1 & \mathbf{0} \\ \ell_1 & I \end{bmatrix} A_{22} \begin{bmatrix} 1 & \ell_1^T \\ \mathbf{0} & I \end{bmatrix}$$

(b) Continue by partitioning $A_{22}^{(2)}$ in a similar way, compute $\widetilde{L}_2$ of order $n-1$ and extend it to L_2 of order n. Continuing, show that after at most $n-2$ steps, a tridiagonal matrix $A^{(r)}$, $r \leq n-1$ has been constructed, such that

$$A^{(r)} = L_{r-1}L_{r-2}\cdots L_1 A L_1^T \cdots L_{r-1}^T.$$

(c) Show that at the sth stage after $s-1$ steps, $1 < s < r$, we have

$$A^{(s)} = \begin{bmatrix} T^{(s-1)} & & \mathbf{0} \\ & & \mathbf{v}^{(s)T} \\ \mathbf{0} & \mathbf{v}^{(s)} & A_{22}^{(s)} \end{bmatrix},$$

where $\mathbf{0}$ is a zero $n-s \times s-1$ matrix, $\mathbf{v} = [a_{s+1,s}^{(s)}, \cdots, a_{ns}^{(s)}]^T$ and $T^{(s-1)}$ is a tridiagonal matrix of order s.

(d) A stabilized version of Aasen's algorithm: For $s = 1, 2, \cdots$ let $\widehat{a}_{a+1,s}^{(s)} = \max_{s+1 \leq i \leq n} |a_{is}^{(s)}|$ and permute the matrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{v}^{(s)T} \\ \mathbf{v}^{(s)} & A_{22}^{(s)} \end{bmatrix}$$

symmetrically to

$$\begin{bmatrix} \mathbf{0} & \widehat{\mathbf{v}}^{(s)T} \\ \widehat{\mathbf{v}}^{(s)} & \widehat{A}_{22}^{(s)} \end{bmatrix},$$

where $\widehat{\mathbf{v}}_{s+1,1}^{(s)} = \widehat{a}_{a+1,s}^{(s)}$. Then perform the factorization of

$$\begin{bmatrix} T^{(s-1)} & & \mathbf{0} \\ & & \widehat{\mathbf{v}}^{(s)T} \\ \mathbf{0} & \widehat{\mathbf{v}}^{(s)} & \widehat{A}_{22}^{(s)} \end{bmatrix}$$

and repeat. Show that all entries $\ell_{ij}^{(s)}$ of L_s satisfy $|\ell_{ij}^{(s)}| \leq 1$.

(e) Implement Aasen's algorithm in Matlab and demonstrate it on a suitable matrix of your choice.

What is the computational complexity of Aasen's algorithm? Compare with that of Gaussian elimination. What are the advantages/disadvantages of Aasen's algorithm compared with the LU factorization?

(f) Historical (optional): Who is Aasen? What is the most relevant article where this algorithm is described.

Note: In general, Aasen's algorithm requires pivoting. However, for this exercise this is not required to be taken into account. If it happens that you have chosen a test matrix for which a zero pivot occurs, please amend the original matrix in a suitable way.

<u>Exercise</u> 2 Let A be a matrix of order $m \times n$. Denote the column vectors of A by $\mathbf{a}_i, i = 1, 2, \cdots, n$.
Consider the Modified Gram-Schmidt algorithm:

```
for i   1 to n
        v_i = a_i
end
for i   = 1 to n
        r_ii = ||v_i||
        q_i = v_i/r_ii
        for j   = i + 1 to n
                r_ij = q_i^* v_j
                v_j = v_j − r_ij q_i
        end
end
```

Each outer step of the Modified Gram-Schmidt algorithm can be interpreted as a right -multiplication by a square upper triangular matrix ($\widehat{R}_i$). For example, beginning with A, the first iteration multiplies the first column $\mathbf{a}_1$ by $1/r_{11}$ and then subtracts r_{1j} times the result of each of the remaining columns $\mathbf{a}_j$. This is equivalent to right-multiplication by a matrix $R_1 (= \widehat{R}_1)$ as follows:

$$\begin{bmatrix} \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ \\ \end{bmatrix} \begin{bmatrix} 1/r_{11} & -r_{12}/r_{11} & -r_{13}/r_{11} & \cdots \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \\ \mathbf{q}_1 & \mathbf{v}_2^{(2)} & \cdots & \mathbf{v}_n^{(2)} \\ \\ \end{bmatrix}$$

At the end of the iteration we have $AR_1R_2 \cdots R_n = Q$. Observe that the matrices $\widehat{R}_i$ are of decreasing order. In order to be able to form the product $R = R_1R_2 \cdots R_n$ we have $R_1 (= \widehat{R}_1)$, $R_2 = \begin{bmatrix} 1 & 0 \\ 0 & \widehat{R}_2 \end{bmatrix}$ etc.

Task: Determine the exact numbers of floating point operations (additions, subtractions, multiplications and divisions involved in computing the factorization $AR = Q$.

<u>Exercise</u> 3 Let $A(n \times n)$ nonsingular be given. We need to form A^{-1} explicitly. Give two algorithms to compute A^{-1} and compare their computational complexity.
Describe possible restrictions on A, related to the numerical stability of the methods you have described.

<u>Exercise</u> 4 Let A be a rectangular matrix of size $(m \times n)$ which has a full rank. Consider the matrix

$$\widetilde{A} = \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}.$$

What is the condition number of the matrix $\widetilde{A}$ in terms of the singular values of A? Hint: Use some proper factorization of $\widetilde{A}$ and the SVD decomposition of A.

<u>Exercise</u> 5 Here is a strip-down of one of the Matlab's built-in m-files:

```
[U,S,V]=svd(A);
S = diag(S);
tol = max(size(A))*S(1)*eps;
r = sum(S>tol);
s = diag(ones(r,1)./S(1:r));
X = V(:,1:r)*s*U(:,1:r)';
```

What does this program compute?

<u>Exercise</u> 6 (Least Squares problem)
In the supplementary directory you find three LS matrices from the Harwell-Boeing collection (`http://math.nist.gov/MatrixMarket/data/Harwell-Boeing/`), `illc1033`, `well1033` and `well1850`, together with the corresponding right-hand-side (rhs) vectors.

`illc1033.mtx` `illc1033_rhs1.mtx`	The matrix `illc1033` of order 1033×320 and a rhs vector
`well1033.mtx` `well1033_rhs1.mtx`	The matrix `well1033` of order 1033×320 and a rhs vector
`well1850.mtx` `well1850_rhs1.mtx`	The matrix `well1850` of order 1850×712 and a rhs vector
`hb_file_read.m`	This is a utility routine which downloads in Matlab a given matrix and its rhs vector. The way to call the utility is as follows: `> [A,b]=hb_file_read('well1033');`

(i) Study the three matrices from the Harwell-Boeing collection. What are their properties?

(ii) Solve one of the above problems. Motivate the choice of your method. Illustrate some disadvantages of some other of the known methods. Can you recommend to use the normal equation approach? Motivate.

<u>Exercise</u> 7 (Weighted Least Squares, Course book, Question 3.4, page 135)
If some components of $A\mathbf{x} - \mathbf{b}$ are more important than others, we can weight them with a scale factor d_i and solve the weighted least Squares problem

$$\min \|D(Ax - b)\|_2$$

instead, where D is a diagonal matrix which contains the (positive) weights d_i. More generally one can consider the problem

$$\min \|Ax - b\|_C, \tag{1}$$

for some symmetric positive definite matrix C. Recall that for any spd matrix C, the scalar product $(C\mathbf{x}, \mathbf{x}) = \mathbf{x}^T C\mathbf{x}$ defines a norm $\|\mathbf{x}\|_C$.
Task: Define the normal equation for problem (1).

<u>Exercise</u> 8 Suppose $\mathbf{x}$ is the Least Squares solution of $A\mathbf{x} = \mathbf{b}$. Form a new matrix B with one additional column which is the sum of the columns of A. Show that if $\mathbf{y}$ is the vector which is formed from $\mathbf{x}$ by adding a zero at the end, then y is a Least Square solution of $B\mathbf{y} = \mathbf{b}$ but is not in the row space of B in general.
(The *row space* of a matrix is the subspace spanned by its row vectors.)

<u>Exercise</u> 9 Two matrices, $A, B \in C^{m\times m}$ are unitary equivalent if there exists a unitary matrix $Q \in C^{m\times m}$, such that $A = QVQ^*$. Is it true or false that A and B are unitary equivalent if and only if they have the same singular values?

<u>Exercise</u> 10 Let$A(m, n)$ be given. Suppose that $B(n, m)$ is obtained by rotating A ninety degrees clockwise on a paper (which is not exactly a standard mathematical transformation, however is performed by `Matlab`'s `rot90(A,-1)`). Do A and B have the same singular values? Prove that the answer is yes or give a counterexample.

<u>Exercise</u> 11 Let $A(m, n)$, $m \geq n$ be given. The following system has to be solved for various values of a parameter p,

$$(A^T A + p\, I_n)\mathbf{x} = A^T\mathbf{b}.$$

- Show that for $p > 0$ and $rank(A) < n$ the matrix $\widetilde{A} \equiv A^T A + p\, I_n$ is invertible.
- Let $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n > 0$ be the nonzero singular values of A and $p > 0$. Show that if $\sigma_n \leq \sigma_1\sqrt{\varepsilon}$, then
$$\varkappa_2(A^T A + pI_n) < \frac{1}{\varepsilon}$$
for $p \geq 0$.

The solutions, in the form of a written report in Swedish or English, should be delivered to me no later than **November 26, 2007**.
Please try to present your arguments as much as if you were writing a scientific report.
Success!

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