Exercise 6 Consider the following well-scaled matrix:

$$W_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 1 \\ -1 & -1 & 1 & \ddots & 0 & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \ddots & 1 & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{bmatrix}$$

(a) Compute the LU-decomposition of W_n without pivoting. Write the result for $n = 5$ and $n = 10$. Determine the element of maximal magnitude that appears during the process as a function of the matrix size n. Can the observed growth cause numerical instabilities when solving systems with the matrix W_n?

The LU factors in the case whete no permutations are done, are found to be

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -1 & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \ddots & 1 & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & 1 & 0 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 2^0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 2 \\ 0 & 0 & 1 & \ddots & 0 & 0 & 2^3 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2^{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 2^{n-1} \end{bmatrix}$$

We solve first $Ly = b$ and then $Ux = y$ and we obtain

$$\begin{aligned} y_1 &= b_1 \\ y_k &= b_k + \sum_{j=1}^{k-1} 2^{k-1-j} b_j, \; k = 2, \cdots, n \end{aligned}$$

and

$$\begin{aligned} x_n &= y_n = 2^{-(n-1)} b_n + \sum_{j=1}^{n-1} 2^{-j} b_j \\ x_k &= y_k + 2^{(k-1)} x_n \\ &= b_k + \sum_{j=1}^{k-1} 2^{k-1-j} b_j - 2^{k-1} \left[2^{-(n-1)} b_n + \sum_{j=1}^{n-1} 2^{-j} b_j \right] \\ &= b_k - 2^{k-n} b_n \end{aligned}$$

One can see that in exact arithmetic the two sums containing the elements of the right-hand-side vector cancel exactly. However, in floating point operations, they will not, accumulating in this way round-off errors in the solution.