Uppsala University
Graduate School in Mathematics and Computing
Institute for Information Technology

Numerical Linear Algebra FMB and MN1
Fall 2007

Mandatory Assignment 1

Below there is a list of 12 exercises. The assignment for Module 1 should consist of six of those and you are allowed to choose exercises from the list depending on your background and particular interest. You are very welcome to do more if you wish. This, however, does not bring any bonuses more than a better understanding of the material.

Exercise 1 (a) Let A be anti-Hermitian, i.e., $A^* = -A$. Prove that the eigenvalues of A are purely imaginary.

(b) Let A be a real skew-symmetric matrix, i.e., $A^T = -A$. If

$$B = (I + A)(I - A)^{-1},$$

show that

$$B^T B = BB^T = I.$$

Exercise 2 If A and B are similarly equivalent, we write $A \sim B$. Show that $\sim$ is an equivalence relation, namely, there hold

(a) $A \sim A$ (reflexivity),

(b) $A \sim B \implies B \sim A$ (symmetry),

(c) $A \sim B$ and $B \sim C \implies A \sim C$ (transitivity).

Exercise 3 Prove that the following matrix is positive definite

$$A = \begin{bmatrix} 1 & 1+i & -1 \\ 1-i & 6 & -3+i \\ -1 & -3-i & 11 \end{bmatrix}$$

(Try to find the most elegant way to do the above.)

Exercise 4 Let A be a square matrix partitioned in a 2-by-2 block form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

let A_{11} and its Schur complement $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ be nonsingular. Show that

$$det(A) = det(A_{11})det(S).$$

Discuss the spectral relations between A, A_{11} and S. Could it be profitable to solve a system $A\mathbf{x} = \mathbf{b}$ via its Schur complement form? Comment on the advantages and disadvantages.

Exercise 5 Claim: Each of the numbers

$$\frac{\|A\|_1}{\|A\|_2}, \frac{\|A\|_\infty}{\|A\|_2}, \frac{\|A\|_\infty}{\|A\|_E}$$

is bounded below by $1/\sqrt{n}$ and above by $\sqrt{n}$. For one of the numbers, on your choice, show that the above property holds true.

Exercise 6 Consider the following well-scaled matrix:

$$W_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 1 \\ -1 & -1 & 1 & \ddots & 0 & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \ddots & 1 & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{bmatrix}$$

(a) Compute the LU-decomposition of W_n without pivoting. Write the result for $n = 5$ and $n = 10$. Determine the element of maximal magnitude that appears during the process as a function of the matrix size n. Can the observed growth cause numerical instabilities when solving systems with the matrix W_n?

(b) Use complete pivoting and repeat part (a). Does this help in some sense? How?

(c) Formulate a statement comparing the results from (a) and (b) and describe the effect this would have when computing the solution of the system $W_n\mathbf{x} = \mathbf{b}$.

Hint: Try on a small-sized matrix first.

Exercise 7 At the first step of the LU factorization of a matrix A we have

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ 0 & & \\ 0 & & \\ \vdots & & A' \\ 0 & & \end{bmatrix}$$

Show that

(i) if A is spd, then A' is also spd;

(ii) if A is diagonally dominant, then A' is also diagonally dominant.

Exercise 8 [Playing with a large matrix] The file `Assign1_matr1.mat` contains a relatively large matrix, which originates from a simulation of a bone tissue, an application from Biomechanics.
Load the matrix in `matlab` via
`load /it/kurs/nlaMN1/public_html/vt06/Assignments/Assign1_matr1.mat`.
The assigned name of the matrix is $A1$. It is symmetric and positive definite. Create a right-hand vector $\mathbf{b}$ which contains one as a first component and the rest of the entries are zeros $\mathbf{b} = [1, 0, 0, \cdots, 0]$.

(a) Examine as much as possible the properties of the given matrix: size, number of nonzero elements, condition number, (some of the) eigenvalues. Is this an ill-conditioned matrix? Why?

(b) Estimate the computer resources you will need in order to compute the LU-factorization of the matrix $A1$.

(c) Try to solve the matrix in `matlab` using any method you can imagine (even a method you have heard of, which has not yet been considered. Can you successfully execute `U=chol(A1)`? Or `x=A1\ b`? Describe the result of your attempts (you should try at least two different ways) - with both positive and negative outcome. Give your comments.

Hints:

1. You may use the matlab command `eigs` to estimate some of the eigenvalues. For instance, `eigs(A1,5,'SM')` will compute an estimate of the five smallest eigenvalues. For more details see `help eigs`.

2. Do not copy the file containing the matrix in your home directories unless you have checked that your disk space is enough.

<u>Exercise</u> 9 Suppose that A is a real matrix of order n and assume that its entries satisfy $|a_{ij}| \leq 1$. Consider the Gauss elimination process with partial pivoting (i.e., pivoting only within the column which is currently eliminated). Show that after k steps of the process, no entry can exceed 2^k.
Note: Exercise 6 demonstrates that in some cases the maximum magnitude can be actually attained.

<u>Exercise</u> 10 Consider the $n \times n$ Householder matrix $H = I - b\mathbf{u}\mathbf{u}^T$, $b = 2/(\mathbf{u}^t\mathbf{u})$. Show that the eigenvalues of H are $\lambda = -1$ and $\lambda = 1$.

<u>Exercise</u> 11 Can a Householder transformation (see Exercise 10) be the identity matrix? Present your arguments.

<u>Exercise</u> 12 The notation $A < B$ should be understood in a positive definite sense, namely, that the matrix $A - B$ is positive definite. Prove the following theorem.
Theorem If $A > B > 0$ then $0 < A^{-1} < B^{-1}$.
Hint: Use some appropriate (inverse) matrix identity.

The solutions (in the form of a written report in Swedish or English) should be delivered to Maya Neytcheva no later than **Nov 13, 2007**.
Success!

Maya (`Maya.Neytcheva@it.uu.se`)