Division of Scientific Computing
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NGSSC: Numerical Methods in Scientific Computing, Spring 2012

Project - Solving the Black-Scholes PDE.

In 1997 Robert C. Merton and Myron S. Scholes received the Nobel Prize in Economy for their method to determine the price of a European call option. In this project we will implement this method using a finite difference scheme and compare it to a Monte Carlo simulation.
The Black-Scholes-Merton model consists of two assets – a bond B and a stock S – with dynamics given by

$$\begin{aligned} dB(t) &= rB(t)dt \\ dS(t) &= \alpha(t,S(t))S(t)dt + \sigma(t,S(t))S(t)dW(t). \end{aligned} \tag{1}$$

Here W is a Wiener process (which is a stochastic process), r is the interest rate, α is the local mean of return of S and σ is the volatility of S.
The owner of a European call option has the right (but not the obligation) to buy a certain stock (=the underlying stock) at a certain time T (=the time of maturity) to a certain price K (=the strike price). As an example we can think of a European call option where the owner has the right to buy an Ericsson B at the price 78 SEK on September 7, 2011.
The problem now is to determine the arbitrage free price (=the price yielding no secure profits in the market) of this option. At the time of maturity it is easy to determine the price. If the price of the underlying stock is less than K, then we will not exercise the option and it is worthless. On the other hand, if the price S of the underlying stock is larger than K, we buy the stock for the price K and sell it for S, giving us a profit of $S-K$. From this we conclude that the price F of the option at time T is

$$F(T) = \Phi(S(T)) = \max(S-K,0). \tag{2}$$

We call the function Φ the contract function.
Black, Merton, and Scholes [1], [3], derived the following partial differential equation that gives the arbitrage free price $F(t,s)$ of an option when the stock price is s at time t

$$\begin{aligned} F_t + rsF_s + 0.5s^2\sigma^2F_{ss} - rF &= 0, \\ F(T,s) &= \Phi(s). \end{aligned} \tag{3}$$

Suitable boundary conditions are

$$\begin{aligned} F(t,0) &= 0, \\ F(t,s_{\max}) &= s_{\max} - K\exp(-r(T-t)), \end{aligned} \tag{4}$$

or

$$\begin{aligned} F_{ss}(t,0) &= 0, \\ F_{ss}(t,s_{\max}) &= 0. \end{aligned} \tag{5}$$

Here $s_{\max}$ denotes the length of the computational interval in space.
An alternative is to simulate the stock price at t using a stochastic differential equation [2], [4]. Let S be the stock price. Then one can show that the equation is

$$dS(t) = rS(t)dt + \sigma S(t)dW(t). \tag{6}$$

The numerical approximation of (6) is using the Euler scheme [2] from time t_n to $t_{n+1} = t_n + \Delta t$

$$S_{n+1} = S_n + rS_n\Delta t + \sigma S_n \Delta W_n. \tag{7}$$

The increment in the Wiener process ΔW in one time step is approximated by $\sqrt{\Delta t}X$, where X is a random variable with normal distribution $\mathcal{N}(0,1)$.
Starting at $t = t_0$ the stock price at $t = T$ is $S(T)$, the solution of (6) at T, which is approximated by S_N with $t_N = T$. Then the value of the option is $\Phi(S_N) = \max(S_N - K, 0)$. Since S_N is a random variable we take the expected value of the option value $E[\Phi(S_N)]$. We have not taken the risk free interest rate into account in (6). Hence, we have to discount the expected value of the option at t_0 and the result is

$$F(t_0, S_0) = \exp(-r(T - t_0))E[\Phi(S_N)]. \tag{8}$$

By computing many trajectories S_{Nj}, $j = 1, \ldots, M$, with different sequences of random numbers, we can determine an approximation of the expected value by computing the average

$$E[\Phi(S_N)] \approx \frac{1}{M}\sum_{j=1}^{M} \Phi(S_{Nj}). \tag{9}$$

The average over M trajectories converges to the expected value when $M \to \infty$ by the law of large numbers.
The **first task** is to solve (3) with (4) or (5) using a finite difference method implemented in MATLAB. Note that you should start at $t = T$ and step backward in time.

- Start by choosing the finite difference method. Motivate your choice of method.
- Implement the method using MATLAB.
- Run the code using $\sigma = 0.3$, $r = 0.05$ and $T = 50$ and let $s_{\max} = 4K$. Present your results in suitable diagrams and graphs.
- Compute a reference solution F_r on a fine grid and with small time steps. Increase the step size Δs by a large factor (> 5) in the s direction and the time step Δt by the same factor and compute the difference between this solution F_1 and the reference solution at $t = 0$. This is the approximate error in F_1. Double Δs and Δt and compute F_2 at $t = 0$ and determine the approximate error in F_2. Do the errors behave in the way you can expect with your discretization?

The **second task** is to solve (6) for M different sequences starting at $S_0 = S(t_0)$ by Euler's method (7) and then compare the convergence of the discounted expected value (8) with the corresponding value from the Black-Scholes equation. How does the convergence depend on M?
You should write a complete report on this project.

Acknowledgment. The project is a modification of a Black-Scholes assignment by Lina von Sydow.

References

[1] F. Black and M. Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy*, 81 (1973), pp. 637–659.

[2] P. E. Kloeden, E. Platen, H. Schurz, *Numerical Solution of SDE Through Computer Experiments*, Springer, Berlin, 1997.

[3] R. C. Merton, The theory of rational option pricing, *Bell Journal of Economics and Management Science*, 4 (1973), pp. 141–183.

[4] Wikipedia entry Black-Scholes, `http://en.wikipedia.org/wiki/Black-Scholes`