Numerical Methods in Scientific Computing

NGSSC

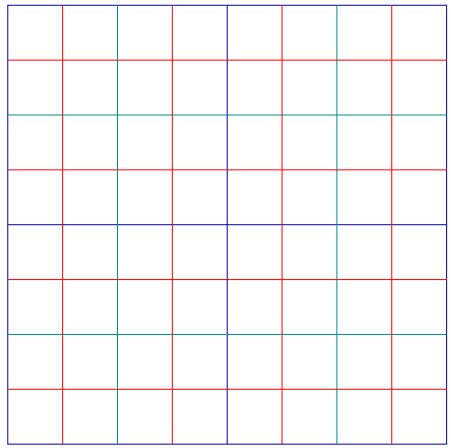
Uppsala University
January 2012

Multigrid/Algebraic Multigrid methods

Run Jacobi demo...

Main idea:

reduce the error $\mathbf{e}^{(k)} = \mathbf{x}_{exact} - \mathbf{x}^{(k)}$ on the given (fine) grid by successive residual corrections on a hierarchy of (nested) coarser grids.



1966, N.S. Bakhvalov and R. Fedorenko

Multilevel preconditioning methods: MG

Procedure MG: $\mathbf{u}^{(k)} \leftarrow MG\left(\mathbf{u}^{(k)}, \mathbf{f}^{(k)}, k, \{\nu_j^{(k)}\}_{j=1}^{k}\right)$;

if $k = 0$, **then** solve $A^{(0)}\mathbf{u}^{(0)} = \mathbf{f}^{(0)}$ exactly or by smoothing,

else

$\mathbf{u}^{(k)} \underset{s_1}{\leftarrow} \mathcal{S}_1^{(k)}\left(\mathbf{u}^{(k)}, \mathbf{f}^{(k)}\right)$, perform s_1 pre-smoothing steps,

Correct the residual:

$\mathbf{r}^{(k)} = A^{(k)}\mathbf{u}^{(k)} - \mathbf{f}^{(k)}$; form the current residual,

$\mathbf{r}^{(k-1)} \leftarrow \mathcal{R}\left(\mathbf{r}^{(k)}\right)$, restrict the residual on the next coarser grid,

$\mathbf{e}^{(k-1)} \leftarrow MG\left(\mathbf{0}, \mathbf{r}^{(k-1)}, k-1, \{\nu_j^{(k-1)}\}_{j=1}^{k-1}\right)$;

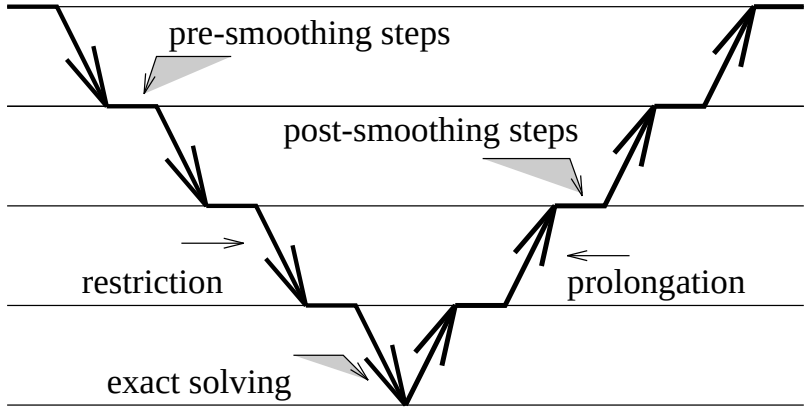
$\mathbf{e}^{(k)} \leftarrow \mathcal{P}\left(\mathbf{e}^{(k-1)}\right)$; prolong the error from the next coarser to the current grid,

$\mathbf{u}^{(k)} = \mathbf{u}^{(k)} - \mathbf{e}^{(k)}$; update the solution,

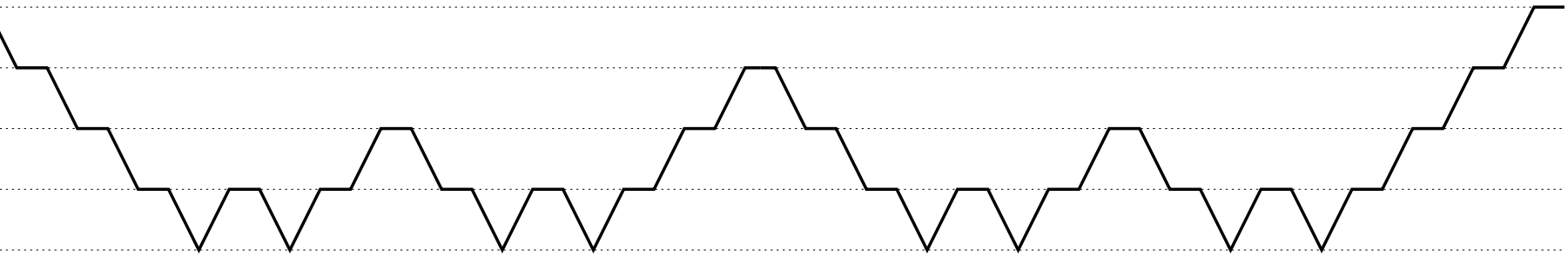
$\mathbf{u}^{(k)} \underset{s_2}{\leftarrow} \mathcal{S}_2^{(k)}\left(\mathbf{u}^{(k)}, \mathbf{f}^{(k)}\right)$, perform s_2 post-smoothing steps.

endif

end Procedure MG



One MG step (V-cycle)



The MG W-cycle

Procedure NI: $\mathbf{u}^{(\ell)} \leftarrow NI\left(\mathbf{u}^{(0)}, \{\mathbf{f}^{(k)}\}_{k=1}^{(\ell)}, \ell, \{\nu^{(k)}\}_{k=1}^{\ell}\right)$;

$\mathbf{u}^{(0)} = {A^{(0)}}^{-1}\mathbf{f}^{(0)}$,

for k= 1 **to** ℓ **do**

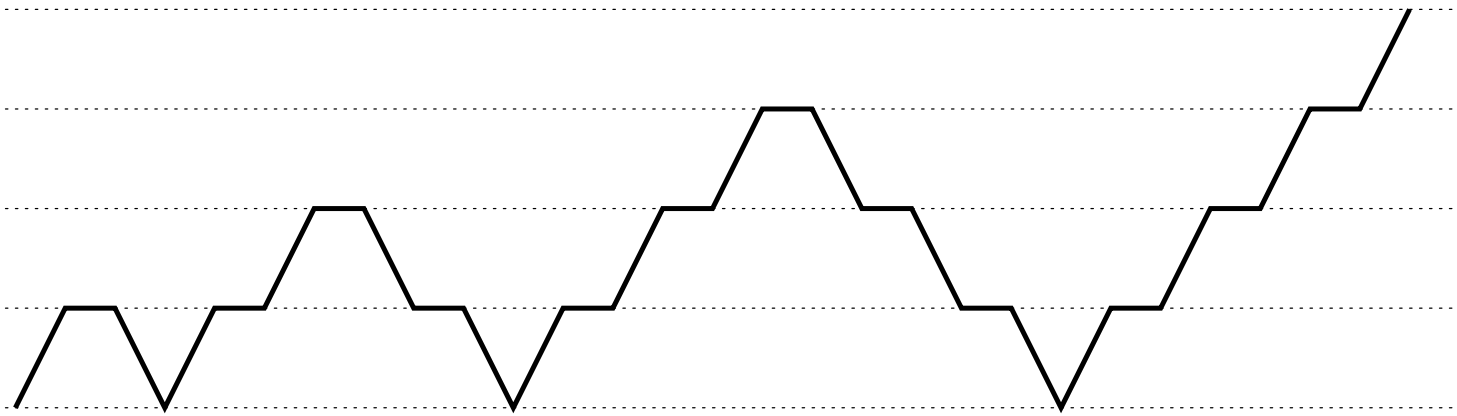
$\mathbf{u}^{(k)} = \mathcal{P}\left(\mathbf{u}^{(k-1)}\right)$;

$\mathbf{u}^{(k)} \leftarrow MG\left(\mathbf{u}^{(k)}, \mathbf{f}^{(k)}, k, \{\nu_j^{(k)}\}_{j=1}^{k}\right)$;

endfor

end Procedure NI

The so-called *full* MG corresponds to **Procedure** $NI(\cdot, \cdot, \ell, \{1, 1, \cdots, 1\})$



The full MG (V-cycle)

A compact formula presenting the MG procedure in terms of a recursively defined iteration matrix:
(i) Let $M^{(0)} = 0$,
(ii) For $k = 1$ to ℓ, define

$$M^{(k)} = {S^{(k)}}^{s_2} \left({A^{(k)}}^{-1} - P_{k-1}^{k} \left(I - {M^{(k-1)}}^{\nu} \right) {A^{(k-1)}}^{-1} R_k^{k-1} \right) A^{(k)} {S^{(k)}}^{s_1},$$

where $S^{(k)}$ is a smoothing iteration matrix (assuming $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same), R_k^{k-1} and P_{k-1}^{k} are matrices which transfer data between two consecutive grids and correspond to the restriction and prolongation operators $\mathcal{R}$ and $\mathcal{P}$, respectively, and $\nu = 1$ and $\nu = 2$ correspond to the V- and W-cycles.

It turns out that in many cases the spectral radius of $M^{(\ell)}$, $\rho\left(M^{(\ell)}\right)$, is independent of ℓ, thus the rate of convergence of the NI method is optimal. Also, a mechanism to make the spectral radius of $M^{(\ell)}$ smaller is to choose s_1 and s_2 larger. The price for the latter is, clearly, a higher computational cost.

Rate of convergence

Multigrid demo ...

Procedure $AMLI$: $\mathbf{u}^{(k)} \leftarrow AMLI\left(\mathbf{f}^{(k)}, k, \nu_k, \{a_j^{(k)}\}_{j=0}^{\nu_k}\right)$;

$[\mathbf{f}_1^{(k)}, \mathbf{f}_2^{(k)}] \leftarrow \mathbf{f}^{(k)}$,

$\mathbf{w}_1^{(k)} = B_{11}^{(k)}\, \mathbf{f}_1^{(k)}$,

$\mathbf{w}_2^{(k)} = \mathbf{f}_2^{(k)} - A_{21}^{(k)}\, \mathbf{w}_1^{(k)}$,

$k = k - 1$,

if $k = 0$ **then** $\mathbf{u}_2^{(0)} = A^{(0)}\, \mathbf{w}_2^{(1)}$, solve on the coarsest level exactly;

else

$\mathbf{u}_2^{(k)} \leftarrow AMLI\left(a_{\nu_k}^{(k)} \mathbf{w}_2^{(k)}, k, \nu_k, \{a_j^{(k)}\}_{j=0}^{\nu_k}\right)$;

for $j = 1$ **to** $\nu_k - 1$:

$\mathbf{u}_2^{(k)} \leftarrow AMLI\left(A^{(k)}\, \mathbf{u}_2^{(k)} + a_{\nu_k - j}^{(k)} \mathbf{w}_2^{(k)}, k, \nu_k, \{a_j^{(k)}\}_{j=0}^{\nu_k}\right)$;

endfor

endif

$k = k + 1$,

$\mathbf{u}_1^{(k)} = \mathbf{w}_1^{(k)} - B_{11}^{(k)} A_{12}^{(k)}\, \mathbf{u}_2^{(k)}$,

$\mathbf{u}^{(k)} \leftarrow [\mathbf{u}_1^{(k)}, \mathbf{u}_2^{(k)}]$

end Procedure $AMLI$