

NGSSC: Numerical Methods in Scientific Computing
January 26, 2012

Assignment: Ordinary differential equations

Requirements: At the end of the lab session, please deliver a Matlab-generated report on the runs you performed. Give your comments and observations (included in the generated report or added by hand).

Where to run Matlab: After logging in on some of the Unix hosts, please issue the command `linuxlogin`. For some more details, see <http://www.it.uu.se/datordrift/maskinpark/linux>.

Exercise 1 (Predator-Prey Dynamics)

The populations of *Canis lupus* (wolves) and *Rangifer tarandus* (reindeer) in the environment can be described by the system of differential equations

$$\begin{aligned} X'(t) &= X(t)(a - bY(t)) \\ Y'(t) &= Y(t)(cX(t) - d), \end{aligned} \tag{1}$$

where $X(t)$ is the population of *Rangifer tarandus* and $Y(t)$ is the population of *Canis lupus*. From the equations we see the following model assumptions. The rate of change (X') of the number of reindeer (X) is proportional to X with a growth factor a minus those, eaten by the wolves. The amount, eaten by the wolves is proportional to the number of wolves (Y), but also to X , thus, the more are the deer the more are eaten. The rate of change (Y') for the wolves is proportional to their number (Y) with a growth factor c minus those who die with a death rate d . For the numerical test, set the following values of the model parameters and initial populations: $a = 8$, $b = .25$, $c = .02$, $d = 20$ $X(0) = 600$ and $Y(0) = 60$. Using the MATLAB code `ode45` solve (1) for $0 \leq T \leq 2$ and plot the solution. Check the details about `ode45`.

Exercise 2 (Model of the heart)

Introduction

A simple model of the heart can be derived from van der Pol's equation which can be written in the form

$$\begin{aligned}\frac{dx}{dt} &= u - \mu \left(\frac{x^3}{3} - x \right) \\ \frac{du}{dt} &= -x\end{aligned}$$

The fluctuation in the length of the heart fibre, as the heart contracts and dilates subject to an electrical stimulus, thus pumping blood through the system, may be represented by this pair of equations.

Starting from the relaxed state, the contractions begin with the application of the the stimulus, slowly at first then becoming faster so giving a sufficient final impetus to the blood. When the stimulus is removed, the heart dilates slowly at first and then more rapidly until it returns to the relaxed state and the cycle can begin again. To follow this behaviour, the van der Pol equation required some modification so that x represents the length of the heart fibre and u can be replaced by a variable that represents the stimulus applied to the heart. This can be done by substituting $s = -u/\mu$, where s represents the stimulus and μ is a constant. Since

$$\frac{ds}{dt} = \frac{-\frac{du}{dt}}{\mu} \implies \frac{du}{dt} = -\mu \frac{ds}{dt}$$

Hence

$$\begin{aligned}\frac{dx}{dt} &= \mu \left(-s - \frac{x^3}{3} + x \right) \\ \frac{ds}{dt} &= \frac{x}{\mu}\end{aligned}$$

However, it is possible to introduce the tension factor p ($p > 0$) into the model. This will take account of the effects of the increased blood pressure in terms of the increased tension in the heart fibre. The final form of the model is then

$$\begin{aligned}\frac{dx}{dt} &= \mu \left(-s - \frac{x^3}{3} + px \right) \\ \frac{ds}{dt} &= \frac{x}{\mu}\end{aligned}$$

Tasks

1. Use the MATLAB `ode23` to simulate the heartbeat. Assume the initial values $x = 0$, $s = -1$ and start the simulation at time $t = 0$. Let the simulation continue until $t = 40$. Use $\mu = 0.5$ and $p = 1$. Use an accuracy tolerance of 10^{-3} . Plot the result and plot the step sizes that have been used by the routine during the solution.
2. What happens when the tension factor is increased to stimulate an increase in blood pressure? Try different values of p , it may be necessary to increase the time interval in order to simulate a complete cycle. What happens when the tension factor is reduced to stimulate a decrease in blood pressure? Try, for example, $p = 0.01$.

3. Use the three MATLAB routines `ode23`, `ode45` and `ode113` when simulating the heart. Let $p = 2$ and stop the simulation when $t = 100$. Use the relative tolerances: 10^{-2} , 10^{-3} , 10^{-4} , 10^{-6} and 10^{-8} . Measure the execution times (using `tic` and `toc` in MATLAB), the number of steps and plot the step size. Which routine is the most efficient? Explain the differences.

The programming in MATLAB should be done as efficiently as possible and so wherever possible try to use vector notation.

Exercise 3 (The mechanics of a variable length pendulum)

Introduction

The equation governing the oscillations of a simple pendulum (a mass m on a string of length r) is

$$\theta'' + \frac{g}{r} \sin(\theta) = 0,$$

where θ is the angle between the pendulum and the vertical. If the length of the pendulum varies with time then the equation becomes

$$\theta'' + 2\frac{r'}{r} + \frac{g}{r} \sin(\theta) = 0.$$

A helicopter is towing a sonar submarine detector through the water. The helicopter rises until the detector is clear of the water. Next, the detector must be winched on board the helicopter without the helicopter changing speed or direction.

Task: 'Kill' the swing



In order to dock successfully, the amplitude of the swing must not exceed $\pi/6$. The cable can be winched in or out at a fixed speed $\alpha m/s$. Assuming that the detector stays clear of the water, it is necessary to devise an algorithm that raises (and lowers) the detector until the maximum amplitude of the swing is less than $\pi/6$ and to raise the detector into the helicopter as quickly as possible. (Note: raising the detector at the maximum amplitude ($\theta = 0$, zero kinetic energy) cannot increase the maximum amplitude, lowering the detector at $\theta = 0$ (maximum kinetic energy, zero potential energy) reduces the maximum amplitude.) For the numerical solution assume $\alpha = \pm 0.5 m/s$, $\theta(0) = 1.0$, $\theta'(0) = 0.0$, $r(0) = 4m$. You should download the MATLAB file `pendel.m` and edit your own strategy for raising and lowering (i.e. define r').

Thanks to Richard White, who originally formulated the problems.