Trigonometric Transforms
Quick Transform Guide

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1 Notation

The symbol i is used for $\sqrt{-1}$, an asterisk * denotes conjugate transposition, an overbar $^-$ denotes conjugation, and a circle $\circ$ means the Hadamard product. For a vector v whose components v_j are indexed like $j = p, p+1, \ldots, q$, the notation $v_{p:q}$ is adopted. The real and imaginary parts of z are designated by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. Finally, the symbol $\texttt{ist}(\cdot)$ would refer to the unnormalized inverse sine transform, whereas its symmetrically normalized counterpart is denoted by the small capitals $\textsc{ist}(\cdot)$. All the transforms throughout this text are similarly abbreviated.

2 Transforms

2.1 The Fourier transform

The entries of the *orthogonal* matrix Q, associated with the inverse Fourier transform (ift), are given by

$$q_{jk} = \exp\Big(\frac{ikj2\pi}{m}\Big), \quad k, j = 0, \ldots, m-1.$$

For the case where m is not a power of two, determine $n = 2^{\lceil \log_2(m-1) \rceil + 1}$, i.e., about twice the nearest larger power of two. Let t, f, and g denote the auxiliary vectors defined by

$$\begin{aligned} t_j &= \exp\Big(\frac{ij^2\pi}{m}\Big), & j &= 0, \ldots, m-1, \\ t_j &\leftarrow 0, & j &= m, \ldots, n-m, \\ t_{n-j} &\leftarrow t_j, & j &= 1, \ldots, m-1, \end{aligned}$$

$$f = \tfrac{1}{n}\texttt{fft}(t),$$

$$g = \tfrac{1}{n}\texttt{fft}(\bar{t}),$$

where fft denotes an ft computed according to a radix-2 FFT algorithm.

2.1.1 The complex case

When m *is* a power of two, the computation $\texttt{ft}(z) \equiv Q^* z$ for $z_{0:m-1} \in \mathbb{C}^m$ could be performed like $\texttt{ft}(z) \leftarrow \texttt{fft}(z)$. When m *is not* a power of two, the *fractional* Fourier transform algorithm is applicable:

$$\begin{aligned} v_j &\leftarrow \bar{t}_j z_j, \quad & j &= 0, \ldots, m-1 \\ v_j &\leftarrow 0, & j &= m, \ldots, n-1 \end{aligned}$$

$$v \leftarrow \texttt{ifft}\big(f \circ \texttt{fft}(v)\big)$$

$$\texttt{ft}(z)_j \leftarrow \bar{t}_j v_j, \quad j = 0, \ldots, m-1$$

For computing $\texttt{ift}(z) \equiv Qz$, the corresponding algorithm is:

$$\begin{aligned} v_j &\leftarrow t_j z_j, \quad & j &= 0, \ldots, m-1 \\ v_j &\leftarrow 0, & j &= m, \ldots, n-1 \end{aligned}$$

$$v \leftarrow \texttt{ifft}\big(g \circ \texttt{fft}(v)\big)$$

$$\texttt{ift}(z)_j \leftarrow t_j v_j, \quad j = 0, \ldots, m-1$$

2.1.2 The real case

In order to compute $\texttt{ft}(x, y)$ for $\{x_{0:m-1}, y_{0:m-1}\} \in \mathbb{R}^m$, i.e., a pairwise $\texttt{ft}$ for real data, the following algorithm is applicable:

$$z \leftarrow x + iy$$

$$f \leftarrow \texttt{ft}(z)$$

$$\begin{aligned} \texttt{ft}(x)_0 &\leftarrow \mathrm{Re}(f_0) \\ \texttt{ft}(y)_0 &\leftarrow \mathrm{Im}(f_0) \\ \begin{aligned} \texttt{ft}(x)_j &\leftarrow \tfrac{1}{2}(\bar{f}_{m-j} + f_j) \\ \texttt{ft}(y)_j &\leftarrow \tfrac{1}{2}i(\bar{f}_{m-j} - f_j) \end{aligned}&, \quad j = 1, \dots, \left\lfloor \frac{m}{2} \right\rfloor \\ \begin{aligned} \texttt{ft}(x)_{m-j} &= \overline{\texttt{ft}(x)_j} \\ \texttt{ft}(y)_{m-j} &= \overline{\texttt{ft}(y)_j} \end{aligned}&, \quad j = 1, \dots, \left\lfloor \frac{m-1}{2} \right\rfloor \end{aligned}$$

For computing $\texttt{ift}(u, v)$ for $\{u_{0:m-1}, v_{0:m-1}\} \in \mathbb{C}^m$ such that $\{u_0, v_0\} \in \mathbb{R}$ and $u_{m-j} = \bar{u}_j$, $v_{m-j} = \bar{v}_j$, $j = 1, \dots, \lfloor \frac{m}{2} \rfloor$, i.e., a pairwise $\texttt{ift}$ for conjugate even data, the corresponding algorithm is:

$$\begin{aligned} z_j &\leftarrow u_j + iv_j, \quad j = 0, \dots, \left\lfloor \frac{m}{2} \right\rfloor \\ z_{m-j} &\leftarrow \bar{u}_j + i\bar{v}_j, \quad j = 1, \dots, \left\lfloor \frac{m-1}{2} \right\rfloor \end{aligned}$$

$$f \leftarrow \texttt{ift}(z)$$

$$\begin{aligned} \texttt{ift}(u) &\leftarrow \mathrm{Re}(f) \\ \texttt{ift}(v) &\leftarrow \mathrm{Im}(f) \end{aligned}$$

2.2 The sine transform

The entries of the unitary matrix Q, associated with the inverse sine transform (IST), are given by

$$q_{jk} = \sqrt{\tfrac{2}{m+1}} \sin\Big(\frac{kj\pi}{m+1}\Big), \quad k, j = 1, \dots, m.$$

Let s denote the auxiliary vector defined by

$$s_j = \sin\Big(\frac{j\pi}{m+1}\Big), \quad j = 1, \dots, m.$$

2.2.1 The real case

In order to compute $\texttt{IST}(x, y) \equiv [Qx, Qy] = [Q^*x, Q^*y] \equiv \texttt{ST}(x, y)$ for $\{x_{1:m}, y_{1:m}\} \in \mathbb{R}^m$, i.e., a pairwise IST or ST for real data, the following algorithm is applicable:

$$\begin{aligned}
u_0 &\leftarrow x_1 \\
v_0 &\leftarrow y_1 \\
\left.\begin{aligned} u_j &\leftarrow \tfrac{1}{2}(x_{2j+1} - x_{2j-1}) - ix_{2j} \\ v_j &\leftarrow \tfrac{1}{2}(y_{2j+1} - y_{2j-1}) - iy_{2j} \end{aligned}\right\}&, \quad j = 1, \dots, \Big\lfloor \frac{m-1}{2} \Big\rfloor \\
u_{\lfloor \frac{m+1}{2} \rfloor} &\leftarrow \begin{cases} -\tfrac{1}{2}x_{m-1} - ix_m & \text{if } m \text{ is even} \\ -x_m & \text{if } m \text{ is odd} \end{cases} \\
v_{\lfloor \frac{m+1}{2} \rfloor} &\leftarrow \begin{cases} -\tfrac{1}{2}y_{m-1} - iy_m & \text{if } m \text{ is even} \\ -y_m & \text{if } m \text{ is odd} \end{cases} \\
\left.\begin{aligned} u_{m+1-j} &= \bar{u}_j \\ v_{m+1-j} &= \bar{v}_j \end{aligned}\right\}&, \quad j = 1, \dots, \Big\lfloor \frac{m}{2} \Big\rfloor
\end{aligned}$$

$$[f, g] \leftarrow \texttt{ift}(u, v)$$

$$\left.\begin{aligned}
\texttt{ST}(x)_j &\leftarrow \tfrac{1}{4}\sqrt{\tfrac{2}{m+1}}\Big(\frac{f_j + f_{m+1-j}}{s_j} + f_j - f_{m+1-j}\Big) \\
\texttt{ST}(y)_j &\leftarrow \tfrac{1}{4}\sqrt{\tfrac{2}{m+1}}\Big(\frac{g_j + g_{m+1-j}}{s_j} + g_j - g_{m+1-j}\Big)
\end{aligned}\right\}, \quad j = 1, \dots, m$$

2.2.2 The complex case

In order to compute $\texttt{IST}(z)$ or $\texttt{ST}(z)$ for $z_{1:m} \in \mathbb{C}^m$, the following algorithm is applicable:

$$
\begin{aligned}
[f, g] &\leftarrow \texttt{ST}\big(\mathrm{Re}(z), \mathrm{Im}(z)\big) \\
\texttt{IST}(z) = \texttt{ST}(z) &\leftarrow f + ig
\end{aligned}
$$

2.3 The cosine transform

The entries of the unitary matrix Q, associated with the inverse cosine transform (ICT), are given by

$$q_{j0} = \begin{cases} \sqrt{\frac{1}{2(m-1)}}, & j = 0 \\ \sqrt{\frac{1}{m-1}}, & j = 1, \ldots, m-2 \\ \sqrt{\frac{1}{2(m-1)}}, & j = m-1 \end{cases}$$

$$q_{j,m-1} = \begin{cases} \sqrt{\frac{1}{2(m-1)}}, & j = 0 \\ \sqrt{\frac{1}{m-1}}(-1)^j, & j = 1, \ldots, m-2 \\ \sqrt{\frac{1}{2(m-1)}}(-1)^{m-1}, & j = m-1 \end{cases}$$

$$q_{jk} = \left\{ \begin{array}{ll} \sqrt{\frac{1}{m-1}}, & j = 0 \\ \sqrt{\frac{2}{m-1}} \cos\left(\frac{kj\pi}{m-1}\right), & j = 1, \ldots, m-2 \\ \sqrt{\frac{1}{m-1}}(-1)^k, & j = m-1 \end{array} \right\}, \quad k = 1, \ldots, m-2.$$

Let s denote the auxiliary vector defined by

$$s_j = \sin\left(\frac{j\pi}{m-1}\right), \quad j = 1, \ldots, m-2.$$

2.3.1 The real case

In order to compute $\texttt{ICT}(x, y) \equiv [Qx, Qy] = [Q^*x, Q^*y] \equiv \texttt{CT}(x, y)$ for $\{x_{0:m-1},\, y_{0:m-1}\} \in \mathbb{R}^m$, i.e., a pairwise ICT or CT for real data, the following algorithm is applicable:

$$
\begin{aligned}
u_0 &\leftarrow \sqrt{2}x_0 \\
v_0 &\leftarrow \sqrt{2}y_0 \\
\left.\begin{aligned}
u_j &\leftarrow x_{2j} + \tfrac{1}{2}i(x_{2j+1} - x_{2j-1}) \\
v_j &\leftarrow y_{2j} + \tfrac{1}{2}i(y_{2j+1} - y_{2j-1})
\end{aligned}\right\}&, \quad j = 1, \ldots, \left\lfloor \frac{m-3}{2} \right\rfloor \\
u_{\lfloor \frac{m-1}{2} \rfloor} &\leftarrow \begin{cases} x_{m-2} + i(\frac{1}{\sqrt{2}}x_{m-1} - \frac{1}{2}x_{m-3}) & \text{if } m \text{ is even} \\ \sqrt{2}x_{m-1} & \text{if } m \text{ is odd} \end{cases} \\
v_{\lfloor \frac{m-1}{2} \rfloor} &\leftarrow \begin{cases} y_{m-2} + i(\frac{1}{\sqrt{2}}y_{m-1} - \frac{1}{2}y_{m-3}) & \text{if } m \text{ is even} \\ \sqrt{2}y_{m-1} & \text{if } m \text{ is odd} \end{cases} \\
\left.\begin{aligned}
u_{m-1-j} &= \bar{u}_j \\
v_{m-1-j} &= \bar{v}_j
\end{aligned}\right\}&, \quad j = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor - 1
\end{aligned}
$$

$$
[f, g] \leftarrow \texttt{ift}(u, v)
$$

$$
\begin{aligned}
\texttt{CT}(x)_0 &\leftarrow \sqrt{\tfrac{1}{m-1}}\Big(\tfrac{1}{\sqrt{2}}(x_0 + x_{m-1}) + \sum_{k=1}^{m-2} x_k\Big) \\
\texttt{CT}(y)_0 &\leftarrow \sqrt{\tfrac{1}{m-1}}\Big(\tfrac{1}{\sqrt{2}}(y_0 + y_{m-1}) + \sum_{k=1}^{m-2} y_k\Big) \\
\texttt{CT}(x)_{m-1} &\leftarrow \sqrt{\tfrac{1}{m-1}}\Big(\tfrac{1}{\sqrt{2}}(x_0 + (-1)^{m-1}x_{m-1}) - \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} x_{2k-1} + \sum_{k=1}^{\lfloor \frac{m-2}{2} \rfloor} x_{2k}\Big) \\
\texttt{CT}(y)_{m-1} &\leftarrow \sqrt{\tfrac{1}{m-1}}\Big(\tfrac{1}{\sqrt{2}}(y_0 + (-1)^{m-1}y_{m-1}) - \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} y_{2k-1} + \sum_{k=1}^{\lfloor \frac{m-2}{2} \rfloor} y_{2k}\Big) \\
\left.\begin{aligned}
\texttt{CT}(x)_j &\leftarrow \tfrac{1}{4}\sqrt{\tfrac{2}{m-1}}\Big(f_j + f_{m-1-j} + \frac{f_j - f_{m-1-j}}{s_j}\Big) \\
\texttt{CT}(y)_j &\leftarrow \tfrac{1}{4}\sqrt{\tfrac{2}{m-1}}\Big(g_j + g_{m-1-j} + \frac{g_j - g_{m-1-j}}{s_j}\Big)
\end{aligned}\right\}&, \quad j = 1, \ldots, m-2
\end{aligned}
$$

2.3.2 The complex case

In order to compute $\texttt{ICT}(z)$ or $\texttt{CT}(z)$ for $z_{0:m-1} \in \mathbb{C}^m$, the following algorithm is applicable:

$$
\begin{aligned}
[f, g] &\leftarrow \texttt{CT}\big(\mathrm{Re}(z), \mathrm{Im}(z)\big) \\
\texttt{ICT}(z) = \texttt{CT}(z) &\leftarrow f + ig
\end{aligned}
$$

2.4 The sine transform-II

The entries of the unitary matrix Q, associated with the inverse sine transform-II (IST2), are given by

$$\left.\begin{aligned} q_{j,k-1} &= \sqrt{\tfrac{2}{m}}\sin\Big(\frac{k(j+\frac{1}{2})\pi}{m}\Big), \quad k=1,\ldots,m-1 \\ q_{j,m-1} &= \sqrt{\tfrac{1}{m}}(-1)^j \end{aligned}\right\}, \quad j=0,\ldots,m-1.$$

Let w denote the auxiliary vector defined by

$$w_j = i\exp\Big(\frac{-ij\pi}{2m}\Big), \quad j=1,\ldots,m-1.$$

2.4.1 The real case

In order to compute $\texttt{ST2}(x,y) \equiv [Q^*x, Q^*y]$ for $\{x_{0:m-1}, y_{0:m-1}\} \in \mathbb{R}^m$, i.e., a pairwise ST2 for real data, the following algorithm is applicable:

$$\begin{aligned} &\left.\begin{aligned} u_j &\leftarrow x_{2j} \\ v_j &\leftarrow y_{2j} \end{aligned}\right., \quad j=0,\ldots,\Big\lfloor\frac{m-1}{2}\Big\rfloor \\ &\left.\begin{aligned} u_{m-j} &\leftarrow -x_{2j-1} \\ v_{m-j} &\leftarrow -y_{2j-1} \end{aligned}\right., \quad j=1,\ldots,\Big\lfloor\frac{m}{2}\Big\rfloor \end{aligned}$$

$$[f,g] \leftarrow \texttt{ft}(u,v)$$

$$\begin{aligned} \texttt{ST2}(x)_{m-1} &\leftarrow \sqrt{\tfrac{1}{m}}\mathrm{Re}(f_0) \\ \texttt{ST2}(y)_{m-1} &\leftarrow \sqrt{\tfrac{1}{m}}\mathrm{Re}(g_0) \\ &\left.\begin{aligned} \texttt{ST2}(x)_{j-1} &\leftarrow \sqrt{\tfrac{2}{m}}\mathrm{Re}(w_j f_j) \\ \texttt{ST2}(y)_{j-1} &\leftarrow \sqrt{\tfrac{2}{m}}\mathrm{Re}(w_j g_j) \end{aligned}\right., \quad j=1,\ldots,\Big\lfloor\frac{m}{2}\Big\rfloor \\ &\left.\begin{aligned} \texttt{ST2}(x)_{m-1-j} &\leftarrow \sqrt{\tfrac{2}{m}}\mathrm{Re}(w_{m-j}\bar{f}_j) \\ \texttt{ST2}(y)_{m-1-j} &\leftarrow \sqrt{\tfrac{2}{m}}\mathrm{Re}(w_{m-j}\bar{g}_j) \end{aligned}\right., \quad j=1,\ldots,\Big\lfloor\frac{m-1}{2}\Big\rfloor \end{aligned}$$

For computing $\texttt{IST2}(x,y) \equiv [Qx, Qy]$ for $\{x_{0:m-1}, y_{0:m-1}\} \in \mathbb{R}^m$, i.e., a pairwise IST2 for real data, the corresponding algorithm is:

$$
\begin{aligned}
u_0 &\leftarrow \tfrac{1}{\sqrt{2}} x_{m-1} \\
v_0 &\leftarrow \tfrac{1}{\sqrt{2}} y_{m-1} \\
\left.\begin{aligned} u_j &\leftarrow \tfrac{1}{2}\bar{w}_j (x_{j-1} + i x_{m-1-j}) \\ v_j &\leftarrow \tfrac{1}{2}\bar{w}_j (y_{j-1} + i y_{m-1-j}) \end{aligned}\right., &\quad j = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \\
\left.\begin{aligned} u_{m-j} &= \bar{u}_j \\ v_{m-j} &= \bar{v}_j \end{aligned}\right., &\quad j = 1, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor
\end{aligned}
$$

$$
[f, g] \leftarrow \texttt{ift}(u, v)
$$

$$
\begin{aligned}
\left.\begin{aligned} \texttt{IST2}(x)_{2j} &\leftarrow \sqrt{\tfrac{2}{m}} f_j \\ \texttt{IST2}(y)_{2j} &\leftarrow \sqrt{\tfrac{2}{m}} g_j \end{aligned}\right., &\quad j = 0, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor \\
\left.\begin{aligned} \texttt{IST2}(x)_{2j-1} &\leftarrow -\sqrt{\tfrac{2}{m}} f_{m-j} \\ \texttt{IST2}(y)_{2j-1} &\leftarrow -\sqrt{\tfrac{2}{m}} g_{m-j} \end{aligned}\right., &\quad j = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor
\end{aligned}
$$

2.4.2 The complex case

In order to compute $\mathtt{ST2}(z)$ for $z_{0:m-1} \in \mathbb{C}^m$, the following algorithm is applicable:

$$\begin{aligned} [f, g] &\leftarrow \mathtt{ST2}\big(\mathrm{Re}(z), \mathrm{Im}(z)\big) \\ \mathtt{ST2}(z) &\leftarrow f + ig \end{aligned}$$

For computing $\mathtt{IST2}(z)$, the corresponding algorithm is:

$$\begin{aligned} [f, g] &\leftarrow \mathtt{IST2}\big(\mathrm{Re}(z), \mathrm{Im}(z)\big) \\ \mathtt{IST2}(z) &\leftarrow f + ig \end{aligned}$$

2.5 The cosine transform-II

The entries of the unitary matrix Q, associated with the inverse cosine transform-II (ICT2), are given by

$$\left.\begin{aligned} q_{j0} &= \sqrt{\tfrac{1}{m}} \\ q_{jk} &= \sqrt{\tfrac{2}{m}}\cos\Big(\frac{k(j+\frac{1}{2})\pi}{m}\Big), \quad k = 1,\ldots,m-1 \end{aligned}\right\}, \quad j = 0,\ldots,m-1.$$

Let w denote the auxiliary vector defined by

$$w_j = \exp\Big(\frac{-ij\pi}{2m}\Big), \quad j = 1,\ldots,m-1.$$

2.5.1 The real case

In order to compute $\texttt{CT2}(x,y) \equiv [Q^*x, Q^*y]$ for $\{x_{0:m-1}, y_{0:m-1}\} \in \mathbb{R}^m$, i.e., a pairwise CT2 for real data, the following algorithm is applicable:

$$\begin{aligned} u_j &\leftarrow x_{2j} \\ v_j &\leftarrow y_{2j} \end{aligned}, \quad j = 0,\ldots,\Big\lfloor\frac{m-1}{2}\Big\rfloor$$

$$\begin{aligned} u_{m-j} &\leftarrow x_{2j-1} \\ v_{m-j} &\leftarrow y_{2j-1} \end{aligned}, \quad j = 1,\ldots,\Big\lfloor\frac{m}{2}\Big\rfloor$$

$$[f,g] \leftarrow \texttt{ft}(u,v)$$

$$\begin{aligned} \texttt{CT2}(x)_0 &\leftarrow \sqrt{\tfrac{1}{m}}\mathrm{Re}(f_0) \\ \texttt{CT2}(y)_0 &\leftarrow \sqrt{\tfrac{1}{m}}\mathrm{Re}(g_0) \end{aligned}$$

$$\begin{aligned} \texttt{CT2}(x)_j &\leftarrow \sqrt{\tfrac{2}{m}}\mathrm{Re}(w_j f_j) \\ \texttt{CT2}(y)_j &\leftarrow \sqrt{\tfrac{2}{m}}\mathrm{Re}(w_j g_j) \end{aligned}, \quad j = 1,\ldots,\Big\lfloor\frac{m}{2}\Big\rfloor$$

$$\begin{aligned} \texttt{CT2}(x)_{m-j} &\leftarrow \sqrt{\tfrac{2}{m}}\mathrm{Re}(w_{m-j}\bar{f}_j) \\ \texttt{CT2}(y)_{m-j} &\leftarrow \sqrt{\tfrac{2}{m}}\mathrm{Re}(w_{m-j}\bar{g}_j) \end{aligned}, \quad j = 1,\ldots,\Big\lfloor\frac{m-1}{2}\Big\rfloor$$

For computing $\texttt{ICT2}(x,y) \equiv [Qx, Qy]$ for $\{x_{0:m-1}, y_{0:m-1}\} \in \mathbb{R}^m$, i.e., a pairwise ICT2 for real data, the corresponding algorithm is:

$$
\begin{aligned}
u_0 &\leftarrow \tfrac{1}{\sqrt{2}} x_0 \\
v_0 &\leftarrow \tfrac{1}{\sqrt{2}} y_0 \\
&\left.\begin{aligned} u_j &\leftarrow \tfrac{1}{2}\bar{w}_j(x_j - i x_{m-j}) \\ v_j &\leftarrow \tfrac{1}{2}\bar{w}_j(y_j - i y_{m-j}) \end{aligned}\right., \quad j = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \\
&\left.\begin{aligned} u_{m-j} &= \bar{u}_j \\ v_{m-j} &= \bar{v}_j \end{aligned}\right., \quad j = 1, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor
\end{aligned}
$$

$$
[f, g] \leftarrow \texttt{ift}(u, v)
$$

$$
\begin{aligned}
&\left.\begin{aligned} \texttt{ICT2}(x)_{2j} &\leftarrow \sqrt{\tfrac{2}{m}} f_j \\ \texttt{ICT2}(y)_{2j} &\leftarrow \sqrt{\tfrac{2}{m}} g_j \end{aligned}\right., \quad j = 0, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor \\
&\left.\begin{aligned} \texttt{ICT2}(x)_{2j-1} &\leftarrow \sqrt{\tfrac{2}{m}} f_{m-j} \\ \texttt{ICT2}(y)_{2j-1} &\leftarrow \sqrt{\tfrac{2}{m}} g_{m-j} \end{aligned}\right., \quad j = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor
\end{aligned}
$$

2.5.2 The complex case

In order to compute $\texttt{CT2}(z)$ for $z_{0:m-1} \in \mathbb{C}^m$, the following algorithm is applicable:

$$\begin{aligned} [f, g] &\leftarrow \texttt{CT2}\big(\mathrm{Re}(z), \mathrm{Im}(z)\big) \\ \texttt{CT2}(z) &\leftarrow f + ig \end{aligned}$$

For computing $\texttt{ICT2}(z)$, the corresponding algorithm is:

$$\begin{aligned} [f, g] &\leftarrow \texttt{ICT2}\big(\mathrm{Re}(z), \mathrm{Im}(z)\big) \\ \texttt{ICT2}(z) &\leftarrow f + ig \end{aligned}$$

2.6 The sine transform-IV

The entries of the unitary matrix Q, associated with the inverse sine transform-IV (IST4), are given by

$$q_{jk} = \sqrt{\tfrac{2}{m}}\sin\Big(\frac{(k+\frac{1}{2})(j+\frac{1}{2})\pi}{m}\Big), \quad k,j = 0,\dots,m-1.$$

Let w and s denote the auxiliary vectors defined by

$$\begin{aligned}
w_j &= \exp\Big(\frac{ij\pi}{2m}\Big), \quad j = 1,\dots,\Big\lfloor\frac{m}{2}\Big\rfloor,\\
s_j &= \sin\Big(\frac{(j+\frac{m+1}{4})\pi}{m}\Big), \quad j = 0,\dots,\Big\lfloor\frac{m-1}{2}\Big\rfloor,\\
s_{m-j} &= \sin\Big(\frac{(j-\frac{m+1}{4})\pi}{m}\Big), \quad j = 1,\dots,\Big\lfloor\frac{m}{2}\Big\rfloor.
\end{aligned}$$

2.6.1 The real case

In order to compute $\texttt{IST4}(x,y) \equiv [Qx, Qy] = [Q^*x, Q^*y] \equiv \texttt{ST4}(x,y)$ for $\{x_{0:m-1}, y_{0:m-1}\} \in \mathbb{R}^m$, i.e., a pairwise IST4 or ST4 for real data, the following algorithm is applicable:

$$\begin{aligned}
&u_0 \leftarrow \tfrac{1}{2}(x_0 + x_{m-1})\\
&v_0 \leftarrow \tfrac{1}{2}(y_0 + y_{m-1})\\
&u_j \leftarrow \tfrac{1}{4}w_j\big(x_j - x_{j-1} + x_{m-j} + x_{m-1-j} - i(x_j + x_{j-1} + x_{m-j} - x_{m-1-j})\big),\\
&v_j \leftarrow \tfrac{1}{4}w_j\big(y_j - y_{j-1} + y_{m-j} + y_{m-1-j} - i(y_j + y_{j-1} + y_{m-j} - y_{m-1-j})\big),\\
&\qquad j = 1,\dots,\Big\lfloor\frac{m}{2}\Big\rfloor
\end{aligned}$$

$$\begin{matrix} u_{m-j} = \bar{u}_j \\ v_{m-j} = \bar{v}_j \end{matrix}, \quad j = 1,\dots,\Big\lfloor\frac{m-1}{2}\Big\rfloor$$

$$[f,g] \leftarrow \texttt{ift}(u,v)$$

$$
\begin{aligned}
\mathtt{ST4}(x)_{\frac{m-1}{2}} &\leftarrow \sqrt{\tfrac{1}{m}}\Big(\sum_{k=0}^{\lfloor\frac{m-1}{4}\rfloor} x_{4k} + \sum_{k=0}^{\lfloor\frac{m-2}{4}\rfloor} x_{4k+1} - \sum_{k=0}^{\lfloor\frac{m-3}{4}\rfloor} x_{4k+2} - \sum_{k=0}^{\lfloor\frac{m-4}{4}\rfloor} x_{4k+3}\Big)\\
\mathtt{ST4}(y)_{\frac{m-1}{2}} &\leftarrow \sqrt{\tfrac{1}{m}}\Big(\sum_{k=0}^{\lfloor\frac{m-1}{4}\rfloor} y_{4k} + \sum_{k=0}^{\lfloor\frac{m-2}{4}\rfloor} y_{4k+1} - \sum_{k=0}^{\lfloor\frac{m-3}{4}\rfloor} y_{4k+2} - \sum_{k=0}^{\lfloor\frac{m-4}{4}\rfloor} y_{4k+3}\Big)\\
\left.\begin{aligned}\mathtt{ST4}(x)_{2j} &\leftarrow \sqrt{\tfrac{1}{m}}\frac{f_j}{s_j}\\ \mathtt{ST4}(y)_{2j} &\leftarrow \sqrt{\tfrac{1}{m}}\frac{g_j}{s_j}\end{aligned}\right., &\quad j = 0, \ldots, \lceil\tfrac{m-1}{4}\rceil - 1, \lfloor\tfrac{m-1}{4}\rfloor + 1, \ldots, \lfloor\tfrac{m-1}{2}\rfloor\\
\left.\begin{aligned}\mathtt{ST4}(x)_{2j-1} &\leftarrow \sqrt{\tfrac{1}{m}}\frac{f_{m-j}}{s_{m-j}}\\ \mathtt{ST4}(y)_{2j-1} &\leftarrow \sqrt{\tfrac{1}{m}}\frac{g_{m-j}}{s_{m-j}}\end{aligned}\right., &\quad j = 1, \ldots, \lceil\tfrac{m+1}{4}\rceil - 1, \lfloor\tfrac{m+1}{4}\rfloor + 1, \ldots, \lfloor\tfrac{m}{2}\rfloor
\end{aligned}
$$

2.6.2 The complex case

In order to compute $\mathtt{IST4}(z)$ or $\mathtt{ST4}(z)$ for $z_{0:m-1} \in \mathbb{C}^m$, the following algorithm is applicable:

$$
\begin{aligned}
[f, g] &\leftarrow \mathtt{ST4}\big(\mathrm{Re}(z), \mathrm{Im}(z)\big)\\
\mathtt{IST4}(z) = \mathtt{ST4}(z) &\leftarrow f + ig
\end{aligned}
$$

2.7 The cosine transform-IV

The entries of the unitary matrix Q, associated with the inverse cosine transform-IV (ICT4), are given by

$$q_{jk} = \sqrt{\tfrac{2}{m}}\cos\Big(\frac{(k+\frac{1}{2})(j+\frac{1}{2})\pi}{m}\Big), \quad k,j = 0,\ldots,m-1.$$

Let w and c denote the auxiliary vectors defined by

$$\begin{aligned} w_j &= \exp\Big(\frac{ij\pi}{2m}\Big), \quad j = 1,\ldots,\Big\lfloor\frac{m}{2}\Big\rfloor, \\ c_j &= \cos\Big(\frac{(j+\frac{m+1}{4})\pi}{m}\Big), \quad j = 0,\ldots,\Big\lfloor\frac{m-1}{2}\Big\rfloor, \\ c_{m-j} &= \cos\Big(\frac{(j-\frac{m+1}{4})\pi}{m}\Big), \quad j = 1,\ldots,\Big\lfloor\frac{m}{2}\Big\rfloor. \end{aligned}$$

2.7.1 The real case

In order to compute $\texttt{ICT4}(x,y) \equiv [Qx, Qy] = [Q^*x, Q^*y] \equiv \texttt{CT4}(x,y)$ for $\{x_{0:m-1}, y_{0:m-1}\} \in \mathbb{R}^m$, i.e., a pairwise ICT4 or CT4 for real data, the following algorithm is applicable:

$$\begin{aligned} u_0 &\leftarrow \tfrac{1}{2}(x_0 - x_{m-1}) \\ v_0 &\leftarrow \tfrac{1}{2}(y_0 - y_{m-1}) \\ u_j &\leftarrow \tfrac{1}{4}w_j\big(x_j + x_{j-1} + x_{m-j} - x_{m-1-j} - i(x_j - x_{j-1} + x_{m-j} + x_{m-1-j})\big), \\ v_j &\leftarrow \tfrac{1}{4}w_j\big(y_j + y_{j-1} + y_{m-j} - y_{m-1-j} - i(y_j - y_{j-1} + y_{m-j} + y_{m-1-j})\big), \\ &\qquad j = 1,\ldots,\Big\lfloor\frac{m}{2}\Big\rfloor \end{aligned}$$

$$\begin{aligned} u_{m-j} &= \bar{u}_j \\ v_{m-j} &= \bar{v}_j \end{aligned}, \quad j = 1,\ldots,\Big\lfloor\frac{m-1}{2}\Big\rfloor$$

$$[f,g] \leftarrow \texttt{ift}(u,v)$$

$$
\begin{aligned}
\texttt{CT4}(x)_{\frac{m-1}{2}} &\leftarrow \sqrt{\tfrac{1}{m}}\Big(\sum_{k=0}^{\lfloor\frac{m-1}{4}\rfloor} x_{4k} - \sum_{k=0}^{\lfloor\frac{m-2}{4}\rfloor} x_{4k+1} - \sum_{k=0}^{\lfloor\frac{m-3}{4}\rfloor} x_{4k+2} + \sum_{k=0}^{\lfloor\frac{m-4}{4}\rfloor} x_{4k+3}\Big)\\
\texttt{CT4}(y)_{\frac{m-1}{2}} &\leftarrow \sqrt{\tfrac{1}{m}}\Big(\sum_{k=0}^{\lfloor\frac{m-1}{4}\rfloor} y_{4k} - \sum_{k=0}^{\lfloor\frac{m-2}{4}\rfloor} y_{4k+1} - \sum_{k=0}^{\lfloor\frac{m-3}{4}\rfloor} y_{4k+2} + \sum_{k=0}^{\lfloor\frac{m-4}{4}\rfloor} y_{4k+3}\Big)\\
\left.\begin{aligned}\texttt{CT4}(x)_{2j} &\leftarrow \sqrt{\tfrac{1}{m}}\frac{f_j}{c_j}\\ \texttt{CT4}(y)_{2j} &\leftarrow \sqrt{\tfrac{1}{m}}\frac{g_j}{c_j}\end{aligned}\right\},&\quad j = 0,\dots,\lceil\tfrac{m-1}{4}\rceil - 1, \lfloor\tfrac{m-1}{4}\rfloor + 1,\dots,\lfloor\tfrac{m-1}{2}\rfloor\\
\left.\begin{aligned}\texttt{CT4}(x)_{2j-1} &\leftarrow \sqrt{\tfrac{1}{m}}\frac{f_{m-j}}{c_{m-j}}\\ \texttt{CT4}(y)_{2j-1} &\leftarrow \sqrt{\tfrac{1}{m}}\frac{g_{m-j}}{c_{m-j}}\end{aligned}\right\},&\quad j = 1,\dots,\lceil\tfrac{m+1}{4}\rceil - 1, \lfloor\tfrac{m+1}{4}\rfloor + 1,\dots,\lfloor\tfrac{m}{2}\rfloor
\end{aligned}
$$

2.7.2 The complex case

In order to compute $\texttt{ICT4}(z)$ or $\texttt{CT4}(z)$ for $z_{0:m-1} \in \mathbb{C}^m$, the following algorithm is applicable:

$$
\begin{aligned}
[f, g] &\leftarrow \texttt{CT4}\big(\mathrm{Re}(z), \mathrm{Im}(z)\big)\\
\texttt{ICT4}(z) = \texttt{CT4}(z) &\leftarrow f + ig
\end{aligned}
$$

2.8 The modified sine transform

The entries of the unitary matrix Q, associated with the inverse modified sine transform (IMST), are given by

$$q_{jk} = \sqrt{\tfrac{2}{m+1}}\, i^{j+1} \sin\Big(\frac{kj\pi}{m+1}\Big), \quad k, j = 1, \ldots, m.$$

Let w denote the auxiliary vector defined by

$$w_j = i^{j+1}, \quad j = 1, \ldots, m.$$

2.8.1 The real case

In order to compute $\texttt{MST}(x, y) \equiv [Q^* x, Q^* y]$ for $\{x_{1:m}, y_{1:m}\} \in \mathbb{R}^m$, i.e., a pairwise MST for real data, the following algorithm is applicable:

$$z_j \leftarrow \bar{w}_j (x_j + i y_j), \quad j = 1, \ldots, m$$

$$f \leftarrow \texttt{ST}(z)$$

$$\begin{array}{l} \texttt{MST}(x)_j \leftarrow \frac{1}{2}(\bar{f}_{m+1-j} + f_j) \\ \texttt{MST}(y)_j \leftarrow \frac{1}{2} i (\bar{f}_{m+1-j} - f_j) \end{array}, \quad j = 1, \ldots, \left\lfloor \frac{m+1}{2} \right\rfloor$$

$$\begin{array}{l} \texttt{MST}(x)_{m+1-j} = \overline{\texttt{MST}(x)}_j \\ \texttt{MST}(y)_{m+1-j} = \overline{\texttt{MST}(y)}_j \end{array}, \quad j = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor$$

For computing $\texttt{IMST}(u,v) \equiv [Qu, Qv]$ for $\{u_{1:m}, v_{1:m}\} \in \mathbb{C}^m$ such that $\bar{u}_j = u_{m+1-j}$, $\bar{v}_j = v_{m+1-j}$, $j = 1, \ldots, \lfloor \frac{m+1}{2} \rfloor$, i.e., a pairwise IMST for conjugate even data, the corresponding algorithm is:

$$
\begin{aligned}
z_j &\leftarrow u_j + i v_j, \quad j = 1, \ldots, \left\lfloor \frac{m+1}{2} \right\rfloor \\
z_{m+1-j} &\leftarrow \bar{u}_j + i \bar{v}_j, \quad j = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor
\end{aligned}
$$

$$
f \leftarrow \texttt{IST}(z)
$$

$$
\begin{aligned}
z_j &\leftarrow w_j f_j, \quad j = 1, \ldots, m \\
\texttt{IMST}(u) &\leftarrow \mathrm{Re}(z) \\
\texttt{IMST}(v) &\leftarrow \mathrm{Im}(z)
\end{aligned}
$$

2.8.2 The complex case

In order to compute $\texttt{MST}(z)$ for $z_{1:m} \in \mathbb{C}^m$, the following algorithm is applicable:

$$v_j \leftarrow \bar{w}_j z_j, \quad j = 1, \ldots, m$$

$$\texttt{MST}(z) \leftarrow \texttt{ST}(v)$$

For computing $\texttt{IMST}(z)$, the corresponding algorithm is:

$$f \leftarrow \texttt{IST}(z)$$

$$\texttt{IMST}(z)_j \leftarrow w_j f_j, \quad j = 1, \ldots, m$$