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Parameterization of State Feedback Gains for Pole Assignment

BY

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May 2001

DEPARTMENT OF SYSTEMS AND CONTROL
INFORMATION TECHNOLOGY
UPPSALA UNIVERSITY
UPPSALA
SWEDEN

Dissertation for the degree of Licentiate of Philosophy in Automatic Control
at Uppsala University 2001

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ISSN 1404-5117
Printed by University Printers, Ekonomikum, Uppsala, Sweden

Abstract

The pole assignment problem has been subject for research for a long time. For single-input single-output systems this problem is well understood but for multi-input multi-output systems the pole assignment problem is more complex. In this thesis a parameterization of state feedback gains for pole assignment is characterized with respect to completeness, redundancy and existence. In order to make a systematic examination of this parameterization a number of classes are introduced.

This parameterization depends on two matrices that can be regarded as design parameters. In the thesis it is shown how the degree of freedom in the pole assignment problem for multi-input systems is characterized by these two matrices.

It turns out that the properties of the parameterization depends on whether the characteristic polynomials of the open and the closed loop systems are coprime or not. It is shown in the thesis that if the characteristic polynomials are coprime, every possible feedback gain can be parameterized in this way, and in this sense the parameterization is complete. If the characteristic polynomials have factors in common the parameterization is not complete. In the thesis the shortcomings of the parameterization for this case are characterized.

The design parameters seem to offer a greater degree of freedom than what can be offered in the pole assignment problem. This indicates a certain degree of overparameterization. This redundancy in the design parameters is characterized in the thesis.

The parameterization implies that a certain matrix is invertible. Necessary conditions for when this matrix is invertible are given in terms of the two design parameters. It is shown that this matrix is invertible for almost every value of the design parameters when the characteristic polynomials are coprime, and hence that the parameterized gains are generally applicable.

The parameterization and its properties are illustrated on a linear model of the military aircraft JAS Gripen.

Acknowledgements

I would like express my sincere gratitude to my supervisor, Dr Kjell Nordström, for his encouragement, guidance and support. He has spent countless hours together with me discussing the essence of automatic control and life in general, and always in a spirit of inspiration, dedication and imagination.

I am very grateful to my colleague, Pär Samuelsson, for his valuable comments on parts of this manuscript, and also for being a very good friend.

Many thanks to everybody at the Department of Systems and Control for being such a good company.

Finally, I send my deepest love and gratitude to Malin for her support, patience and endless love.

Contents

Chapter 1

Introduction

In the design of linear control systems the concept of poles is of fundamental importance since their location characterizes the dynamic properties of the system. Stability, speed and damping are properties that are highly dependent on the position of the poles of the system.
When linear feedback control is applied, the position of the poles are almost always modified, regardless of the used design technique.
One design approach is based on pole assignment (pole placement, eigenvalue assignment), where the objective is to obtain some desired closed loop system poles. In standard textbooks of automatic control the pole assignment design technique for single-input single-output systems is presented in both the polynomial and the state space formalism.
This is mainly because there is a diversity of quite simple techniques for pole assignment of SISO systems. The role of the pole position for the dynamical system and its performance is well understood.
For multi-input multi-output (MIMO) systems the problem of directing poles into a certain prespecified region is more difficult. The optimal linear quadratic design method (LQ) is often proposed. This method is based on the formulation of a quadratic criterion which in most cases is chosen in an ad hoc manner.
Textbooks recommend that the weights in the performance index are iteratively guessed in order to obtain a closed loop system with good sensitivity and robustness and acceptable poles. This fact has motivated for a vast development in design methods that are optimized for robustness and sensitivity. The so called H_∞ based controllers are an outcome of this work.
Regardless of which performance index the user wants to optimize, the structure of the control law often can be implemented as an observer based state feedback control law. The role of the observer is to recover incomplete state information and it can be regarded as the dual of the state feedback control problem.
This thesis deals with the link between the closed loop poles and the realization of the state feedback gain matrix. Since there is a close connection between this and the dual problem of examining the state observer, the study made in the

thesis is also applicable for the state observer case.
While there is a one to one correspondence between the closed loop poles and the feedback gain for a single-input system, there is a whole class of feedback gains for which the desired poles are obtained for multi-input systems. This degree of freedom can be used to develop algorithms with additional design specifications. Such an approach limits the design to the state feedback problem. That is, all state variables are assumed to be available.
In a broader context, *i.e.*, when both an observer and a state feedback gain are present in the controller, the design specifications can be completely different as have been indicated above. For instance, robustness and sensitivity may be far more relevant design objectives than separated design objectives for the observer and the state feedback.
This motivates for the need to develop closed form expressions for the feedback gains. In the thesis such a parameterization is considered.
The parameterization depends on the open loop system quantities and the desired closed loop poles in an intuitively clear way. The degree of freedom is expressed in terms of a matrix which is of the same dimension as the state feedback gain.
The parameterization is a direct generalization of Ackermann's formula, which is one way to express the one to one correspondence between the closed loop poles and the gain for single-input systems.

Outline of the thesis

The pole assignment problem has been a subject of research for a long time. A brief background on some different approaches to pole assignment is given in Chapter 2.
In Chapter 3 preliminaries for the rest of this thesis are introduced. The state feedback gain parameterization is presented and discussed to some extent.
In order to make a systematic examination of the parameterization a number of classes are introduced in Chapter 3. These classes and the introduced notations are used throughout the thesis. Thus, Chapter 3 is the basis for the rest of the thesis.
The Sylvester equation turns out to be very useful in the characterization of the parameterization. A discussion on how the parameterized gains are related to the Sylvester equation is given in Chapter 4.
In Chapter 5 the completeness of the parameterization is discussed. It is shown that for total pole assignment, *i.e.*, when all poles are moved, all possible feedback gains can be parameterized in this way. It is also shown that this is not the case for partial pole assignment, *i.e.*, when some poles do not change.
The parameterization depends on two parameters. The first of these is a square matrix of the same dimension as the order of the system. The second parameter

is a matrix of the same dimension as the state feedback gain. Both of these may be regarded as design parameters. A discussion on how these two parameters affect the closed loop system is also given in Chapter 5. Redundancy in these parameters is also treated here.
Chapter 6 deals with the question of existence. The parameterization contains the inverse of a matrix, so the existence of any parameterized feedback gain depends on the existence of this inverse. This matrix depends on the two design parameters mentioned above. The main result of Chapter 6 is that for total pole assignment this matrix is nonsingular for almost every value of the design parameters. It is also shown that, for certain open loop systems, this matrix is singular regardless of the values of the parameters, depending on if certain poles should remain unchanged. Necessary conditions on the design parameters for nonsingularity of this matrix are also given in the chapter.
In Chapter 7 some of the results presented in the thesis are illustrated on a linear model of the Swedish military aircraft JAS Gripen. The model represents the turning dynamics in the horizontal plane. It is also illustrated how the parameterization can be used for controller design in combination with linear quadratic design.

Chapter 2

Background

The problem addressed in this thesis is to find state feedback gain matrices for a linear time-invariant system, so that the poles of the closed loop system coincides with some predefined complex numbers. This problem formulation may seem to be applicable only for a very special and idealized situation. For a number of reasons one may ask whether this kind of study has any relevance. For instance, almost no "real" system is linear and time-invariant, pure state feedback is almost never applicable since the state is rarely available for measurement, all models are more or less erratic (there are always uncertainties), and so on. Still, the theory of linear systems, time-invariant ones in particular, is fundamental for most control applications. This is mainly because many systems can be approximated with a linear time-invariant model (around some stationary point). Although the model is an approximation in these cases, the linear theory can be used to predict the behaviour of the real system to a large extent. Thus the theory of linear time-invariant systems is justified.

In the sequel the discussion is confined to finite dimensional linear time-invariant models (and controllers) with a state space realisation. The pole location is of fundamental importance for the behaviour of a linear system. The stability of a system depends on whether all poles lie within the stability region or not. Even for stable systems, the pole location has vital importance for the behaviour of the system in terms of speed and relative damping. A commonly used approach for feedback controller design is pole assignment, which is motivated by this fact. The desired pole location can for instance be based on knowledge of the nature of the system.

A common controller structure is the observer based state feedback control law. This is in line with the certainty equivalence principle, *i.e.*, the control problem is divided into two separate subproblems. The first is to solve a deterministic control problem where the state is known, and the second is a state estimation problem. This approach is successfully used in LQG control, for which the separation theorem guarantees that the optimal controller is obtained (for some presumptions). For the state estimation an observer is used, and the problem to obtain an observer has many similarities to the state feedback control problem. In fact, these problems can be regarded as each others dual. By solving one problem, the dual problem is in principle solved as well. In practice, though, both problems should be taken into account when designing the controller.

The observer based state feedback control structure is used in for example LQG, as mentioned above, and H_∞-control (see for instance [55] or [47]). It is also very suitable for the pole assignment approach mentioned above, which is naturally separated into one pole assignment problem for the closed loop system and one for the observer. Both of these subproblems have in principle the same solution. This explains why the discussion in this thesis is confined to the state feedback control problem. The discussion is valid, with minor and obvious modifications, to the observer pole assignment problem.

No matter what approach is used for the controller design, the use of observer based state feedback control structure can always be interpreted in terms of pole assignment. For instance, for a given state feedback gain, one might ask if and how the state feedback gain could be obtained with a pole assignment technique. Another aspect is that of combining pole assignment with some other approach, for instance LQG. For this kind of issues it would be benificial to have some parameterization for the state feedback gain based on the closed loop system poles.

The pole assignment problem is classic and has been studied thouroughly within linear system theory. A fundamental result, shown by Wonham 1967 [53], is that a necessary and sufficient condition for arbitrary pole assignment with state feedback is that the system is controllable. For systems with only one input the pole assignment problem is quite straightforward — a specific set of desired closed loop system poles determines uniquely the state feedback gain, and there are some closed expressions for the feedback gain (see for instance Chapter 3.2.1 in [25]). However, for multi-input systems a whole class of feedback gains yields identical closed loop poles. This induces an additional degree of freedom in the choice of the feedback gain. There is a number of algorithms for determining the feedback gain for the pole assignment problem, with different approaches to making use of this degree of freedom. In many textbooks (for instance [46] and [25]) the sufficiency of controllability for arbitrary pole assignment is shown with a constructive proof, which can be used as an algorithm for the pole assignment problem. However, this involves a state transformation to the canonical controller form ([34]), and possibly an input transformation. There are straightforward, but rather cumbersome, recipes (see [25] or [46]) for how to find this state transformation, including the search for independent columns in the controllability matrix. While the canonical controller shows the structure of the open loop system well, it gives no hint of what happens to the closed system in the original, untransformed state space. Also, the natural choice of state feedback gain for the controller form yields a closed loop system in companion form. It is not obvious how the feedback gain should be chosen to obtain another form. There are more involved algorithms based on the controller form (see for instance [50]) or other canonical forms, like the Jordan form (see for instance [19]).

Another common approach in textbooks is that of transforming the multi-input system to a single-input system, and then simply use a formula for the unique (since the system now is single-input) state feedback gain. This can be achieved by choosing a linear combination of the available inputs and then use this linear combination as a single input. A consequence of this approach is that the control/input signal is confined to have the same direction all the time. This direction is defined by the choice of linear combination of the inputs. That is, the input is confined to $\mathbb{R}$ instead of $\mathbb{R}^m$. Furthermore, the obtained single-input system must be controllable, a fact that is crucial for the choice of linear combination of the inputs. In fact, there are controllable systems for which no linear combination of the inputs will yield a controllable single-input system. This situation can be avoided with the aid of Heymann's lemma [22]. That is, a "new" open loop system is obtained by a preliminary state feedback with an arbitrary (full rank) feedback gain matrix. For almost every such feedback gain, the "new" system obtained is controllable for some linear combination of the inputs. Then the single-input technique described above is applicable to this new system. Apart from being a two-step method, the inclusion of the arbitrary preliminary state feedback obscure the clarity of the otherwise neat single-input formulas.

This approach is also overparameterized. If the system is of order n, and has m inputs, the preliminary state feedback involves nm (in practice) free parameters, and the choice of linear combination of the inputs adds m extra parameters. Thus, in total there are $(n+1)m$ free parameters. However, the feedback gain obtained has nm entries. Of these, intuitively n should depend on the n desired poles of the closed loop system, leaving $(n-1)m$ degrees of freedom. And still, in spite of this overparameterization, not all possible feedback gains can be obtained in this way. This is due to the fact that the last step is a single-input state feedback. State feedback preserves controllability, so the obtained closed loop system will be controllable for some linear combination of the inputs (it will be single-input controllable). However, systems with multiple poles need not be single-input controllable. Such closed loop systems cannot be obtained with the proposed method.

A third approach to the pole assignment problem is to use the solution of the Sylvester equation. The Sylvester equation, which is discussed in Chapter 4, is a linear matrix equation. This approach has been proposed by, amongst others, Bhattacharyya and de Souza in 1982 [8], but it has been used frequently before that as well (see for instance [35] and [51]). The Sylvester equation occurs quite naturally when the inverse eigenvalue problem is studied. In fact, every state feedback gain can be obtained from the solution of a Sylvester equation. It has the appealing feature that the system matrices can be used as they are, and essentially no further knowledge about the system is needed. A drawback with this approach is that the Sylvester equation does not always have a solution, or the solution may not be unique. It is also required that the solution is invertible. Bhattacharyya and de Souza [8] avoid these problems by only considering total pole assignment, *i.e.*, the case when the characteristic polynomials of the open and closed loop system are coprime. They also confine themselves to feedback gains that yield single-input controllable closed loop systems (see above).

There are many proposed approaches for how to make use of the degree of freedom that appear in the multi-input case. One of these concerns the numerical aspects. This is often referred to as robust pole assignment. Due to limited word length and ill conditioned matrices the poles of the closed loop system can differ from the desired poles. The objective is often to minimize the discrepancy between the desired an the obtained closed loop system poles. Many have utilized the Sylvester equation approach mentioned above for this purpose; see for instance Bhattacharyya and co-authors [9], [10] and [27], Kautsky *et al.* 1985 [26], Tits and Yang 1996 [49], Datta and Hetti 1997 [12] and Lam *et al.* [32] and [33]. There are other approaches to the robust pole assignment problem as well, like the QR related algorithms presented in Miminis and Paige 1982 [38], [37] and 1988 [39].

Another way to utilize the nonuniqueness of the pole assignment problem is to combine it with some other controller synthesis approach. Optimal control, in particular LQ control, is one approach that frequently has been interlaced with the pole assignment problem. The objective is then to find a feedback gain that simultanuously yields some desired closed loop system poles, and minimizes some quadratic performance index. A number of techniques has been used for this purpose. One is to solve the pole assignment problem approximately, as in Anderson and Moore 1971 (Chapter 5.4) [5], or to see that the poles end up in some preassigned region, like in Amin 1985 [4], Wittenmark *et al.* 1987 [52] or Haddad and Bernstein 1992 [20]. Another technique is to assign some (not all) of the closed loop system poles exactly and let the remaining poles depend on some optimal solution, like in Anderson and Moore 1971 (Chapters 5.4 and 7.4) [5], Maki and Van de Vegte 1974 [36], Sugimoto 1998 [48] or Iracleous and Alexandridis 1999 [24]. Others have manipulated the performance index matrices, as Bar-Ness 1978 [7], or approached the problem by assigning the eigenvectors as well as the eigenvalues/poles, see Alexandridis and Galanos 1987 [2], Alexandridis 1996 [1] or Choi and Seo 1999 [11].

In Moore 1976 [40] it is shown showed how the eigenvalues/poles and the eigenvectors of the closed loop system matrix (*i.e.*, $A - BL$) can modified with state feedback (at least if the eigenvalues are distinct). An outline of this method is given in Chapter 5 in this thesis. The problem of assigning the eigenvalues and eigenvectors simultanuously is often referred to as the eigenstructure assignment problem. It has been studied by, for instance Klein and Moore 1977 [31], Porter and D'Azzo 1978 [44], Fahmy and O'Reilly 1982 [17] and Askarpour and Owens

1999 [6].
An issue related to pole assignment via state feedback is the problem of pole assignment via output feedback. This problem has been studied by, among others, Kimura 1975 [28], 1977 [29], 1994 [30], Davison and Wang 1975 [13], Paraskevopoulos and King 1976 [43], Alexandridis and Paraskevopoulos 1996 [3] and Eisinberg *et al.* 1999 [16].

Chapter 3

Preliminaries

This Chapter contains the preliminaries to the remaining chapters of the thesis. In Section 3.1 the state feedback pole assignment problem is formulated. Section 3.1 also introduces some general notations and terminology. Then Section 3.2 introduces notations specific for this thesis. The main result of this chapter, a parameterization of state feedback gains, is presented in Section 3.3. This parameterization was first presented in Nordström and Norlander 1997 [41]. In this thesis the properties of this parameterization are analyzed and characterized. In order to clarify the discussion and analysis of these properties, a number of classes are introduced in Section 3.4. These classes are used extensively in the remaining chapters of the thesis.

3.1 Problem formulation

Consider the continuous-time multi-input linear time-invariant system with the dynamic state equation

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) \tag{3.1}$$

where the vectors $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ for every fixed t, and the constant matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, where $m > 1$. The presentation in this thesis is based on the continuous-time system (3.1), but as the discussion mainly concerns algebraic properties the discussion is valid for discrete-time systems as well. Without loss of generality, the matrix B is assumed to have full (column) rank.

Of fundamental importance for the behaviour of the system (3.1) is the location of its poles, *i.e.*, the eigenvalues of the matrix A. For instance, the location of the poles determines entirely whether the system (3.1) is asymptotically stable, stable or unstable. Therefore it is desirable to be able to modify the location of the poles using state feedback control.

$$u(t) = -Lx(t) + w(t), \quad L \in \mathbb{R}^{m\times n} \tag{3.2}$$

Here the matrix L is the *feedback gain* and $w(t) \in \mathbb{R}^m$ represents the new input to the system. The latter can be used to enter a reference signal into the system. A reference signal is usually associated with an output, but here the full state is of more interest. Therefore $w(t) = 0$ will be assumed.

The system (3.1) will be referred to as the *open loop system*. The poles of the open loop system are the zeros of the characteristic polynomial

$$a(s) \triangleq \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n \tag{3.3}$$

where $a_i \in \mathbb{R}$, for $i = 1, \ldots, n$, and $s \in \mathbb{C}$. The choice of s as variable in (3.3) is motivated by its association to the Laplace transform. For discrete-time systems it is replaced by z (the Z-transform formalism).

Applying the state feedback control law (3.2) to (3.1) yields ($w(t) = 0$ is assumed)

$$\frac{d}{dt}x(t) = (A - BL)x(t) \tag{3.4}$$

which will be referred to as the *closed loop system*. The purpose with the control is to see that the poles of the closed loop system coincide with n predefined complex numbers, closed under complex conjugation. These complex numbers are the desired poles of the closed loop system. An equivalent way to say this is that the closed loop system should have a predefined (desired) characteristic polynomial. Let

$$p(s) \triangleq s^n + p_1 s^{n-1} + \cdots + p_n \tag{3.5}$$

where $p_i \in \mathbb{R}$, for $i = 1, \ldots, n$, be the desired characteristic polynomial of the closed loop system. The problem is now to find a feedback gain matrix L so that

$$\det(sI - A + BL) = p(s)$$

In the single input case ($m = 1$) the feedback gain, which is a row vector, is uniquely determined by $p(s)$, and there are several analytic expressions for it, for instance the formulas due to Ackermann and Bass, Gura [25]. In the multi-input case, however, the feedback gain is not uniquely determined by $p(s)$, and none of the mentioned formulas is applicable. In what follows, an analytic parameterization of the feedback gain for the multi-input case will be given. For convenience, though, some notations will first be made.

3.2 Notations

The characteristic polynomials $a(s)$ and $p(s)$ will be associated to vectors whose entries are the coefficients of the polynomials respectively:

$$\alpha \triangleq \begin{bmatrix} a_n \\ \vdots \\ a_1 \end{bmatrix}, \quad \pi \triangleq \begin{bmatrix} p_n \\ \vdots \\ p_1 \end{bmatrix}, \quad \alpha, \pi \in \mathbb{R}^n \tag{3.6}$$

Compare with equations (3.3) and (3.5) and note the reversed order of the coefficients in α and π.
To further abbreviate in notations, the standard ON-base vectors in $\mathbb{R}^n$ will be denoted e_i for $i = 1, \dots, n$, *i.e.*, a 1 at the ith entry and zeros elsewhere. Especially $e_n = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T$ will be used frequently. The square $(n \times n)$ matrix, with ones in all subdiagonal entries and zeros elsewhere, will be denoted I_L. Except for this, I will always denote the identity matrix and an index marks the size of it (for instance, I_m is the $m \times m$ identity matrix).
Let the coefficients of the polynomial

$$r(s) = s^n + r_1 s^{n-1} + \cdots + r_n$$

be stacked into the vector $\rho = \begin{bmatrix} r_n & \cdots & r_1 \end{bmatrix}^T$. To the polynomial $r(s)$ the *right companion matrix*

$$K_\rho \triangleq \begin{bmatrix} 0 & & \cdots & & 0 & -r_n \\ 1 & 0 & & & & \\ 0 & 1 & 0 & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \\ 0 & \cdots & 0 & 1 & 0 & -r_2 \\ 0 & \cdots & 0 & 0 & 1 & -r_1 \end{bmatrix} \tag{3.7}$$

is associated. Then $K_\rho = I_L - \rho e_n^T$. An important property of K_ρ is that

$$\det(sI - K_\rho) = r(s) \tag{3.8}$$

Thus $K_\alpha = I_L - \alpha e_n^T$ and $K_\pi = I_L - \pi e_n^T$ are the right companion matrices associated to $a(s)$ and $p(s)$ respectively.
A fundamental result (Wonham 1967 [53]) is that an arbitrary characteristic polynomial for the closed loop system can be achieved by state feedback control if and only if the open loop system is controllable, *i.e.*, if and only if the controllability matrix

$$\mathcal{C} \triangleq \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \tag{3.9}$$

have full (row) rank. Therefore the open loop system (3.1) is assumed to be controllable throughout this treatise.
Polynomials with a matrix in the argument will be used in the sequel. For instance, if the polynomial $p(\cdot)$, defined in Equation (3.5), has the matrix A as the argument, every s^i in (3.5) is replaced by A^i for $i = 0, 1, \ldots, n$, so

$$p(A) = A^n + p_1 A^{n-1} + \cdots + p_n I \tag{3.10}$$

since $A^0 = I$.
There are two useful properties associated to the matrices and vectors defined above. These properties involve the *Kronecker product*. The Kronecker product is denoted with $\otimes$, and is defined as

$$F \otimes H \triangleq \begin{bmatrix} f_{11}H & \ldots & f_{1n}H \\ \vdots & & \vdots \\ f_{m1}H & \ldots & f_{mn}H \end{bmatrix} \quad \text{where} \quad F = \begin{bmatrix} f_{11} & \ldots & f_{1n} \\ \vdots & & \vdots \\ f_{m1} & \ldots & f_{mn} \end{bmatrix}$$

for any two matrices F and H.
Using this we have

Property 3.1

$$p(A)B = \mathcal{C}\left[(\pi - \alpha) \otimes I_m\right] \tag{3.11}$$

and

Property 3.2

$$A\mathcal{C} = \mathcal{C}\left[K_\alpha \otimes I_m\right] \tag{3.12}$$

Note that

$$A^n = -a_1 A^{n-1} - a_2 A^{n-2} - \cdots - a_n I = -\begin{bmatrix} I & A & \ldots & A^{n-1} \end{bmatrix} \begin{bmatrix} \alpha \otimes I \end{bmatrix}$$

according to Caley-Hamilton's theorem. Hence

$$p(A)B = A^n B + \mathcal{C}\left[\pi \otimes I_m\right] = \mathcal{C}\left[\pi \otimes I_m\right] - \mathcal{C}\left[\alpha \otimes I_m\right]$$

from which Property 3.1 follows. Also,

$$A\mathcal{C} = \mathcal{C}\left[I_L \otimes I_m\right] + A^n B\left[e_n^T \otimes I_m\right] = \mathcal{C}\left[I_L \otimes I_m\right] - \mathcal{C}\left[\alpha e_n^T \otimes I_m\right]$$

which leads to Property 3.2.

3.3 The parameterized state feedback gains

With the aid of the notation introduced above, the main result of this chapter now can be formulated, *i.e.*, an explicit expression of the feedback gain. This was first presented in Nordström and Norlander 1997 [41].

Theorem 3.1 ([41]) *Let $p(s)$ be a given monic polynomial, M a matrix such that* $\det(sI - M) = p(s)$, *and let Γ satisfy the equation*

$$\left[K_\pi \otimes I_m\right] \Gamma = \Gamma M \tag{3.13}$$

If the product $\mathcal{C}\Gamma$ is nonsingular, then

$$\det(sI - A + BL) = p(s) \tag{3.14}$$

with A and B as in Equation (3.1) *and*

$$L = \left[e_n^T \otimes I_m\right] \Gamma(\mathcal{C}\Gamma)^{-1} p(A) \tag{3.15}$$

The theorem states that the closed loop system has the desired poles under the mentioned assumptions. The feedback gain is thus given by Equation (3.15).

Proof Assume that $\mathcal{C}\Gamma$ is nonsingular. Using Equation (3.15) in the left hand side of (3.14) yields

$$\begin{gathered}\det(sI - A + BL) = \det(sI - A + B\left[e_n^T \otimes I_m\right] \Gamma(\mathcal{C}\Gamma)^{-1} p(A)) = \\ \det(sI - A + p(A)B\left[e_n^T \otimes I_m\right] \Gamma(\mathcal{C}\Gamma)^{-1})\end{gathered}$$

Now Property 3.1 and Property 3.2 can be used:

$$\begin{gathered}\det(sI - A + p(A)B\left[e_n^T \otimes I_m\right] \Gamma(\mathcal{C}\Gamma)^{-1}) = \\ \det(s\mathcal{C}\Gamma - \mathcal{C}\left[K_\alpha \otimes I_m\right]\Gamma + \mathcal{C}\left[(\pi - \alpha)e_n^T \otimes I_m\right]\Gamma) \det(\mathcal{C}\Gamma)^{-1} = \\ \det(s\mathcal{C}\Gamma - \mathcal{C}\left[K_\pi \otimes I_m\right]\Gamma) \det(\mathcal{C}\Gamma)^{-1}\end{gathered}$$

Now, by the use of (3.13),

$$\begin{gathered}\det(s\mathcal{C}\Gamma - \mathcal{C}\left[K_\pi \otimes I_m\right]\Gamma) \det(\mathcal{C}\Gamma)^{-1} = \\ \det(s\mathcal{C}\Gamma - \mathcal{C}\Gamma M) \det(\mathcal{C}\Gamma)^{-1} = \det(sI - M) = p(s)\end{gathered}$$

which completes the proof. ■

To justify that Equation (3.15) truly is a closed form expression for the feedback gain, the following lemma states an explicit expression for the matrix Γ.

Lemma 3.2 ([41]) *All solutions to the identity*

$$\left[K_\pi \otimes I_m\right]\Gamma = \Gamma M \tag{3.16}$$

are

$$\Gamma = \begin{bmatrix} \Gamma_0 \chi_{n-1}(M) \\ \vdots \\ \Gamma_0 \chi_1(M) \\ \Gamma_0 \end{bmatrix} \tag{3.17}$$

where $\chi_j(\cdot)$, $j = 1, \ldots, n-1$ *are polynomials defined as*

$$\chi_j(s) = s^j + \sum_{k=1}^{j} p_k s^{j-k} \tag{3.18}$$

and where Γ_0 *is an arbitrary* $m \times n$*-matrix.*

Remark An equivalent definition of the polynomials in Equation (3.18) is the recursive formula

$$\chi_0(s) = 1, \quad \chi_j(s) = s\chi_{j-1}(s) + p_j, \quad j = 1, \ldots, n-1 \tag{3.19}$$

Proof The result can be derived directly by studying

$$\Gamma = \begin{bmatrix} \Gamma_{n-1} \\ \vdots \\ \Gamma_0 \end{bmatrix}$$

where $\Gamma_j \in \mathbb{R}^{m\times n}$ for $j = 0, \ldots, n-1$. With Γ partitioned like this the left hand side of (3.16) is

$$\left[I_L \otimes I_m\right] \begin{bmatrix} \Gamma_{n-1} \\ \vdots \\ \Gamma_0 \end{bmatrix} - \left[\pi e_n^T \otimes I_m\right] \begin{bmatrix} \Gamma_{n-1} \\ \vdots \\ \Gamma_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \Gamma_{n-1} \\ \vdots \\ \Gamma_1 \end{bmatrix} - \begin{bmatrix} p_n\Gamma_0 \\ p_{n-1}\Gamma_0 \\ \vdots \\ p_1\Gamma_0 \end{bmatrix}$$

while the right hand side of (3.16) reads

$$\Gamma M = \begin{bmatrix} \Gamma_{n-1}M \\ \vdots \\ \Gamma_0 M \end{bmatrix}$$

Together this gives $\Gamma_j = \Gamma_{j-1}M + p_j\Gamma_0$, for $j = 1, \dots, n-1$, and $0 = \Gamma_{n-1}M + p_n\Gamma_0$. This implies that

$$\Gamma_j = \Gamma_0(M^j + \sum_{k=1}^{j} p_k M^{j-k}) = \Gamma_0\chi_j(M), \quad \text{for} \quad j = 1, \dots, n-1$$

and $0 = \Gamma_0 p(M)$. But the last condition is always satisfied since $p(M) = 0$ according to Caley-Hamilton's theorem. ■

Remark In the single input case, where $m = 1$, Γ_0 will be a row vector and Γ will be a square $n \times n$ matrix, which will be nonsingular for an appropriate choice of Γ_0 and M. Furthermore, the controllability matrix will be a square $n \times n$ matrix, which is nonsingular since the open loop system is assumed to be controllable. Thus Equation (3.15) turns to

$$L = e_n^T \Gamma \left(\mathcal{C}\Gamma\right)^{-1} p(A) = e_n^T \mathcal{C}^{-1} p(A) \tag{3.20}$$

which is recognised as Ackermann's formula (see for instance Kailath [25] Chapter 3.2).

Applying Lemma 3.2 to Equation (3.15) yields the somewhat more convenient expression

$$L = \Gamma_0(\mathcal{C}\Gamma)^{-1}p(A) \tag{3.21}$$

When the results in Theorem 3.1 and Lemma 3.2 originally were presented in [41], the right companion matrix K_π was used instead of M in the right hand sides of Equations (3.13) and (3.16). A possible choice that works in most cases is of course to set $M = K_\pi$. However, by allowing any M satisfying $\det(sI - M) = p(s)$, the results in Theorem 3.1 and Lemma 3.2 become more general.

The parameterization in Theorem 3.1 and Lemma 3.2 give rise to a number of questions. In the forthcoming chapters some of these questions are thoroughly discussed. For instance, how efficient is the parameterization? That is, does the parameterization cover all possible feedback gains? If not, which feedback gains can, or cannot, be parameterized in this way? These questions are discussed in Chapter 5.

The matrices M and Γ_0 in the feedback gain parameterization may both be regarded as design parameters. The restriction $\det(sI - M) = p(s)$ perhaps suggests that M should be chosen in a first turn, leaving Γ_0 as the free design parameter. In Section 5.1 it is shown that this is a reasonable procedure. The question then is how Γ_0 should be chosen. The parameterization relies on the

existence of the inverse $(\mathcal{C}\Gamma)^{-1}$, so a first matter of interest is to choose Γ_0 so that $\mathcal{C}\Gamma$ is nonsingular. This is a nontrivial issue. In Chapter 6 the existence of $(\mathcal{C}\Gamma)^{-1}$ is discussed, and in Section 6.1 some necessary conditions for the existence are presented. As mentioned above, the feedback gain yielding the desires closed loop system poles is not unique — it is rather a whole class of feedback gains. In Section 5.2 there are results showing how this degree of freedom is reflected by Γ_0. Thus Γ_0 can be used to exploit this degree of freedom, for instance by forming optimization problems, as is illustrated in Chapter 7.
The following example is to illustrate the use of Theorem 3.1 and Lemma 3.2.

Example 3.1 Consider the continuous-time system

$$\frac{d}{dt}x = Ax + Bu, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

of which all poles are in the origin. The system is controllable and the controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose that it is desired to place all the closed loop poles in -1, so that the desired characteristic polynomial is

$$p(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$$

and that M is chosen as the associated right companion matrix

$$M = K_\pi = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$

The polynomials in Equation (3.18) then are $\chi_1(s) = s+3$ and $\chi_2(s) = s^2+3s+3$. This gives

$$\chi_1(M) = M + 3I = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 3 & -3 \\ 0 & 1 & 0 \end{bmatrix}, \quad \chi_2(M) = M^2 + 3M + 3I = \begin{bmatrix} 3 & -1 & 0 \\ 3 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

When applying Lemma 3.2 with

$$\Gamma_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

the matrix Γ becomes

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

and thus

$$\mathcal{C}\Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Incidently $(\mathcal{C}\Gamma)^{-1} = \mathcal{C}\Gamma$ for this choice of Γ_0. Furthermore, $A^2 = 0$ so

$$p(A) = A^3 + 3A^2 + 3A + I = 3A + I = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying Equation (3.21) then gives the feedback gain

$$L = \Gamma_0(\mathcal{C}\Gamma)^{-1}p(A) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -3 & 3 \end{bmatrix}$$

and with the control law $u = -Lx$ the closed loop system becomes

$$\frac{d}{dt}x = (A - BL)x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 3 & -3 \end{bmatrix} x$$

It is easily verified that the closed loop system has all poles in -1. $\circ$

3.4 Definitions of classes

In the forthcoming chapters the parameterization in Theorem 3.1 and Lemma 3.2 will be analyzed. As an aid for this and to simplify the presentation of the analysis, a number of classes will be defined.

For all definitions below the presumption is that a controllable open loop system (3.1) is given. The open loop system has a characteristic polynomial given by (3.3). Furthermore, it is presumed that a desired characteristic polynomial (3.5)

for the closed loop system is given, and that state feedback in accordance with (3.2) shall be used. Any matrix denoted M in the definitions is presumed to satisfy $\det(sI - M) = p(s)$.
The first set of classes concern the pole assignment problem in general. In order to stress this fact, some of these classes have π as a subscript[1]. This subscript should not be interpreted as an argument, but rather as a mark that they are defined under the presumptions made above. The classes with argument (within parentheses) are defined with respect to their argument.

Definition 3.1 $\mathfrak{L}_\pi$ is the class of all feedback gains that yield the desired poles of the closed loop system, *i.e.*,

$$\mathfrak{L}_\pi = \{L \in \mathbb{R}^{m\times n} \,|\, \det(sI - A + BL) = p(s)\} \tag{3.22}$$

○

Definition 3.2 $\mathfrak{M}(L)$ is the class of all square matrices that are similar (see Definition A.1 in Appendix A) to the closed loop system matrix $A_c = A - BL$ for a fixed feedback gain L, *i.e.*,

$$\mathfrak{M}(L) = \{M \in \mathbb{R}^{n\times n} | \exists T \in \mathbb{R}^{n\times n},\ T^{-1}(A - BL)T = M\} \tag{3.23}$$

○

Definition 3.3 $\mathfrak{M}_\pi$ is the class of all square matrices that are similar to any possible closed loop system matrix $A_c = A - BL$ where the feedback gain matrix belongs to $\mathfrak{L}_\pi$, *i.e.*,

$$\mathfrak{M}_\pi = \bigcup_{L\in\mathfrak{L}_\pi} \mathfrak{M}(L) \tag{3.24}$$

○

Definition 3.4 $\mathfrak{L}(M)$ is the class of all feedback gain matrices that make the closed loop system matrix $A_c = A - BL$ to be similar to M, *i.e.*,

$$\mathfrak{L}(M) = \{L \in \mathbb{R}^{m\times n} | \exists T \in \mathbb{R}^{n\times n},\ T^{-1}(A - BL)T = M\} \tag{3.25}$$

○

[1] The vector π, as defined in (3.6), determines the desired characteristic polynomial uniquely as $p(s) = \begin{bmatrix} 1 & s & \dots & s^{n-1} \end{bmatrix} \pi + s^n$.

The next set of classes are defined under the same presumptions as the previous classes. However, the classes concern the parameterization of the feedback gain that is given by Theorem 3.1 and Lemma 3.2. To stress this, some of these classes have Γ as subscript to distinguish them from the more general classes previously defined. Again, this subscript should not be interpreted as an argument, and classes with argument (within parantheses) are defined with respect to their argument.

Definition 3.5 $\mathfrak{G}(M)$ is the class of all matrices $\Gamma_0 \in \mathbb{R}^{m\times n}$ that, when applied in Lemma 3.2, yields nonsingular matrices $\mathfrak{C}\Gamma$ for a specific matrix M (satisfying $\det(sI - M) = p(s)$), *i.e.*,

$$\mathfrak{G}(M) = \{\Gamma_0 \in \mathbb{R}^{m\times n} \,|\, \det(\mathfrak{C}\Gamma) \neq 0\} \tag{3.26}$$

○

Definition 3.6 $\mathfrak{L}_\Gamma(M)$ is the class of all feedback gain matrices that can be parameterized as in Theorem 3.1 and Lemma 3.2 with a certain matrix M (satisfying $\det(sI - M) = p(s)$), *i.e.*

$$\mathfrak{L}_\Gamma(M) = \{L \in \mathbb{R}^{m\times n} \,|\, L = \Gamma_0(\mathfrak{C}\Gamma)^{-1}p(A), \left[K_\pi \otimes I_m\right]\Gamma = \Gamma M, \Gamma_0 \in \mathfrak{G}(M)\} \tag{3.27}$$

○

Definition 3.7 $\mathfrak{L}_\Gamma$ is the class of all feedback gain matrices that can be parameterized as in Theorem 3.1 and Lemma 3.2, *i.e.*,

$$\mathfrak{L}_\Gamma = \bigcup_M \mathfrak{L}_\Gamma(M) \tag{3.28}$$

○

For the last definitions, the presumptions made above are somewhat relaxed, since they concern the desired poles/characteristic polynomial of the closed loop system. Still, it is presumed that a controllable open loop system (3.1) is given. An important issue in the analysis will be the coprimeness between $a(s)$ and $p(s)$, *i.e.*, the question whether any of the desired poles of the closed loop system is also a pole of the open loop system. To handle this we introduce the following class:

Definition 3.8 $\mathfrak{P}_{cop}$ is the class of all π so that $p(s)$ and $a(s)$ are coprime, *i.e.*,

$$\mathfrak{P}_{cop} = \{\pi \in \mathbb{R}^n \,|\, \nexists s \in \mathbb{C}, p(s) = a(s) = 0\} \tag{3.29}$$

○

It turns out that the class $\mathfrak{P}_{cop}$ is crucial in the characterization of the parameterization in Theorem 3.1 and Lemma 3.2. In order to establish this the following definition will be frequently used in the sequel.

Definition 3.9 The case when $\pi \in \mathfrak{P}_{cop}$ is the *total pole assignment problem*, and the case when $\pi \notin \mathfrak{P}_{cop}$ is the *partial pole assignment problem.* $\circ$

Under the presumption of Definitions 3.1–3.9 $A - BL$ will have the desired characteristic polynomial $p(s)$ if and only if $L \in \mathfrak{L}_\pi$. Since the system is assumed to be controllable and multi-input, $\mathfrak{L}_\pi$ contains more than one element for every $\pi \in \mathbb{R}^n$. The class $\mathfrak{L}(M)$ is another way of classifying the feedback gains. It is clear that $\mathfrak{L}(M)$ is a subclass of $\mathfrak{L}_\pi$ under the presumptions made above.
Obviously $\det(sI - M) = p(s)$ for all $M \in \mathfrak{M}_\pi$. However, the converse is not always true, *i.e.*, $\det(sI - M) = p(s)$ does not imply that $M \in \mathfrak{M}_\pi$. It is true when all zeros of $p(s)$ are distinct. But if $p(s)$ has multiple zeros there may exist matrices M that does not belong to $\mathfrak{M}_\pi$. In Section 5.2 it is shown that the M in the right hand side of Equation (3.13) in Theorem 3.1 is confined to $\mathfrak{M}_\pi$, at least for the total pole assignment problem. The class $\mathfrak{M}_\pi$ is characterized by Rosenbrock's control structure theorem (Theorem B.7 in Appendix B), and is further discussed in Section 5.1.
For the classes $\mathfrak{G}(M)$ and $\mathfrak{L}_\Gamma(M)$ it is assumed that $\det(sI - M) = p(s)$ in accordance with Theorem 3.1. Whenever Γ_0 is picked from $\mathfrak{G}(M)$ (if nonempty) it is merely assumed that $\mathfrak{C}\Gamma$ is nonsingular for this particular choice of M and Γ_0. As has been mentioned above, both M and Γ_0 can be regarded as design parameters in the feedback gain parameterization. If M is fixed, the class $\mathfrak{L}_\Gamma(M)$ is the class of feedback gains that is generated by the parameterization with Γ_0 as the only free parameter. Then, for the class $\mathfrak{L}_\Gamma$, M is also allowed to vary, and thus $\mathfrak{L}_\Gamma$ covers all feedback gains that can be parameterized in this way. Clearly $\mathfrak{L}_\Gamma(M)$ is a subset of $\mathfrak{L}_\Gamma$, which in turn is a subset of $\mathfrak{L}_\pi$ (according to Theorem 3.1).
In the sequel the above classes in the characterization of the gains parameterized in Theorem 3.1 and Lemma 3.2. Before this is done we will interpret the parameterized gains in terms of the Sylvester equation.

Chapter 4

State feedback and the Sylvester equation

In this chapter the Sylvester equation and its significance for the pole assignment problem is discussed. In Section 4.1 it is shown how the Sylvester equation can be interpreted in terms of similarity transformation, and how solutions of Sylvester equations can be used to obtain feedback gains. In Section 4.2 some results are presented. These results show that components of the parameterization in Section 3.3 satisfies Sylvester equations. Together with the similarity transformation interpretation in Section 4.1 the results in Section 4.2 are used to a great extent in the analysis in Chapters 5 and 6.

4.1 General properties

The *Sylvester equation* is a linear matrix equation, which can be written as

$$FX - XG = H \tag{4.1}$$

where F and G are square matrices, not necessarly of the same dimension, H is a matrix of compatible dimension, and X, the unknown, has the same dimension as H. It can be shown (see Theorem A.4 in Appendix A) that if F and G have no common eigenvalues, the Sylvester equation (4.1) has a unique solution. Otherwise, if F and G have common eigenvalues, either there is no solution, or the solution is not unique (see Theorem A.4). The Sylvester equation may be regarded as a generalisation of the Lyapunov equation, but the opposite view is perhaps more relevant, *i.e.*, the Lyapunov equation is a special case of the Sylvester equation.

The Sylvester equation has been used frequently in linear control theory. Particulary in the theory for observers it appears frequently, see for instance Luenberger [35] and O'Reilly [42]. In 1982 Bhattacharyya and de Souza [8] the Sylvester equation is used to discuss the total pole assignment problem.

The idea behind this approach is to say that the closed loop system matrix, $A - BL$ in Equation (3.4), should be similar to a matrix M, which have the desired eigenvalues, *i.e.*,

$$X^{-1}(A - BL)X = M \quad \Longleftrightarrow \quad AX - XM = BLX \tag{4.2}$$

and then try to find the similarity transformation matrix X. Clearly M should be chosen from $\mathfrak{M}_\pi$ (see Definition 3.3). In the expression above there are two unknowns, L and X, so in order to be able to solve for both of these, the identity $LX = G$ is introduced for some arbitrary matrix $G \in \mathbb{R}^{m \times n}$. Then (4.2) turns into

$$AX - XM = BG \tag{4.3}$$

which is recognised as a Sylvester equation. With G given, Equation (4.3) can be solved for X, and then

$$L = GX^{-1} \tag{4.4}$$

provided that X is nonsingular. The idea is thus very simple, but there is no guarantee that it works. As mentioned above, Equation (4.3) has a unique solution if and only if the matrices A and M have no eigenvalues in common. In order to guarantee the existence of a solution of (4.3) M must therefore not have any eigenvalue in common with A. This means that, if no further precaution is taken due to this fact, this approach is only applicable for the total pole assignment problem. This is of course a drawback for an otherwise neat and simple approach. (At a first glance it seems that the only information needed to apply the method is the matrices A and B, as they are.) Another drawback of this approach is that it requires the solution X of (4.3) to be nonsingular (not only to be able to solve for L, but the approach relies on similarity and X is the similarity transformation matrix). Whether X is nonsingular or not depends on the matrices A, M, B and G. In the pole assignment problem the matrices A and B are given by the open loop system, and usually it is assumed that (A, B) is controllable. Under these assumptions, and with M chosen from the class $\mathfrak{M}_\pi$, as defined in Definition 3.3, there exist feedback matrices L and nonsingular matrices X transforming $A - BL$ into M. Thus, if $M \in \mathfrak{M}_\pi$, the nonsingularity of X is due to the choice of G. How G should be chosen in order to guarantee a nonsingular solution of (4.3) is a non-trivial problem; see for instance de Souza and Bhattacharyya [14] and Hearon [21] for some necessary conditions for nonsingularity of the solution X. This will also be discussed in Chapter 6 of this thesis.

That M belongs to $\mathfrak{M}_\pi$ is obviously a necessary condition for the existence of a nonsingular solution of 4.3, since the approach relies on similarity. Bhattacharyya and de Souza [8] avoids the problem of the existence of a nonsingular solution by confining themselves to the total pole assignment problem, and by restricting M to be cyclic[1]. Kautsky, Nichols and van Dooren [26] also start from the Sylvester equation (Problem 2 in [26]), but they set M to be diagonal. This give rise to problems in the case when there are multiple poles among the desired poles of the closed loop system. This follows since diagonal matrices not necessarily belong to $\mathfrak{M}_\pi$.

4.2 The parameterized feedback gains as a solution to the Sylvester equation

The method described in Chapter 3 may also be viewed as a way of using the solution to a Sylvester equation; the difference is that the solution is known and can be used explicitly. This can be seen from the following lemma.

Lemma 4.1 *The product $\mathcal{C}\Gamma$ is a solution to the Sylvester equation in X*

$$AX - XM = p(A)B\Gamma_0 \tag{4.5}$$

with matrices defined as in Chapter 3.

Proof By applying $\mathcal{C}\Gamma$ to the left hand side of (4.5) and using Equation (3.12) and (3.13) it follows that

$$A\mathcal{C}\Gamma - \mathcal{C}\Gamma M = \mathcal{C}\left[K_\alpha \otimes I_m\right]\Gamma - \mathcal{C}\left[K_\pi \otimes I_m\right]\Gamma =$$
$$\mathcal{C}\left[(\pi-\alpha)e_n^T \otimes I_m\right]\Gamma = \mathcal{C}\left[(\pi-\alpha)\otimes I_m\right]\left[e_n^T \otimes I_m\right]\Gamma$$

The second last identity follows from the definition of right companion matrices in (3.7). Then, according to Equation (3.11)

$$\mathcal{C}\left[(\pi-\alpha)\otimes I_m\right]\left[e_n^T \otimes I_m\right]\Gamma = p(A)B\left[e_n^T \otimes I_m\right]\Gamma = p(A)B\Gamma_0$$

which is equal to the right hand side of (4.5). ■

Equation (4.5) is somewhat different from (4.3) by the premultiplication with $p(A)$ on the right hand side of the equation. However, Lemma 4.1 guarantees the existence of a solution to (4.5), no matter what the eigenvalues of A and M are, *i.e.*, the eigenvalues of M may coincide with some of the eigenvalues of A. Though, if A and M have common eigenvalues, (4.5) will not have a unique solution (see Theorem A.4). Still, any nonsingular solution to (4.5) will do. To see this, study the characteristic polynomial of the closed loop system. Equation (3.21) suggests that the feedback gain matrix should be chosen as

[1] All cyclic matrices belong to $\mathfrak{M}_\pi$.

$L = \Gamma_0 X^{-1} p(A)$, where X is any such nonsingular solution. The characteristic polynomial will thus be

$$\det(sI - A + BL) = \det(sI - A - B\Gamma_0 X^{-1} p(A)) = \det(sI - A + p(A)B\Gamma_0 X^{-1}) = \det(sI - A + (AX - XM)X^{-1}) = \det(sI - XMX^{-1}) = \det(sI - M)$$

which is the desired characteristic polynomial for the closed loop system.
It turns out that, when A and M have no eigenvalues in common, Equation (4.5) can be transformed into the form of Equation (4.3), which is stated in the following corollary.

Corollary 4.2 ([41]) *Assume that $\pi \in \mathfrak{P}_{cop}$, then the product $(p(A))^{-1}\mathfrak{C}\Gamma$ is the unique solution to the Sylvester equation*

$$A\Omega - \Omega M = B\Gamma_0 \tag{4.6}$$

Proof The corollary follows from Lemma 4.1 and Theorem A.4. Note that $p(A)$ is nonsingular, and that A and M have no common eigenvalues, since $\pi \in \mathfrak{P}_{cop}$. ■

From (4.6) it is seen that Γ_0 has the same role as G in (4.3) when $\pi \in \mathfrak{P}_{cop}$. Since $(p(A))^{-1}$ is nonsingular for this case, the matrix $\mathfrak{C}\Gamma$ is nonsingular exactly when the solution to (4.6) is nonsingular. Thus the necessary conditions for nonsingularity in de Souza and Bhattacharyya [14] mentioned above are applicable. In Chapter 6.1 some necessary conditions for the nonsingularity of $\mathfrak{C}\Gamma$ are presented. The role of the matrices M and Γ_0 is further discussed in Chapter 5.1.

Chapter 5

Completeness and redundancy of the parameterization

In this chapter the class of feedback gain matrices $\mathfrak{L}_\Gamma$ is considered, *i.e.*, feedback gain matrices generated according to Theorem 3.1 and Lemma 3.2. An important issue is how extensive the class $\mathfrak{L}_\Gamma$ is. If the parameterization in Theorem 3.1 is to be utilized in analysis or design it should preferably cover all feedback gains, *i.e.*, $\mathfrak{L}_\Gamma = \mathfrak{L}_\pi$ is desirable.
It turns out that it is crucial in the study of $\mathfrak{L}_\Gamma$ if the pole assignment is total or partial, *i.e.*, whether all poles of the open loop system should be modified ($\pi \in \mathfrak{P}_{cop}$), or if some open loop poles should be kept so that $p(s)$ and $a(s)$ have common zeros ($\pi \notin \mathfrak{P}_{cop}$). In Section 5.2 the class $\mathfrak{L}_\Gamma$ is characterized for the total pole assignment problem. The main result of Section 5.2 is that $\mathfrak{L}_\Gamma$ cover all possible feedback gains for $\pi \in \mathfrak{P}_{cop}$. In Section 5.3 it is shown that, while the parameterization still is applicable, the class $\mathfrak{L}_\Gamma$ does not contain all feedback gains.
If the matrix M is fixed and Γ_0 is regarded as a free parameter, the same feedback gain can be generated with different choices of Γ_0. This redundancy in Γ_0 is discussed and characterized in Section 5.4.
In most of the discussions contained in this thesis, the two parameters M and Γ_0 are treated somewhat different. According to Theorem 3.1 M is restricted by the condition $\det(sI - M) = p(s)$, while Γ_0 is (almost) completely free. Therefore M is mainly assumed to have been chosen in an initial step, and is assumed to be constant. Thus Γ_0 is the only free parameter. This is a reasonable approach as is shown in Section 5.1, where it also is discussed how M characterize the closed loop system.

5.1 The role of the explicit design parameters

The parameterization presented in Chapter 3 seems to offer two design parameters, Γ_0 and M. To examine the role of these, consider any similarity transformation of M, *i.e.*, let $M_2 = TM_1T^{-1}$ for an arbitrary nonsingular matrix T. Assume that $L \in \mathfrak{L}_\Gamma(M_1)$ and that $L = \Gamma_0(\mathfrak{C}\Gamma)^{-1}p(A)$ in accordance with Equation (3.21) for some $\Gamma_0 \in \mathfrak{G}(M_1)$. The classes $\mathfrak{L}_\Gamma(M)$ and $\mathfrak{G}(M)$ are defined in Definitions 3.6 and 3.5 respectively. Here it is understood that M_1 is used in the parameterization of L. Then $L = \Gamma_0T^{-1}(\mathfrak{C}\Gamma T^{-1})^{-1}p(A)$, and Equation (3.17) gives

$$\Gamma T^{-1} = \begin{bmatrix} \Gamma_0\chi_{n-1}(M_1) \\ \vdots \\ \Gamma_0\chi_1(M_1) \\ \Gamma_0 \end{bmatrix} T^{-1} =$$

$$\begin{bmatrix} \Gamma_0\chi_{n-1}(T^{-1}M_2T)T^{-1} \\ \vdots \\ \Gamma_0\chi_1(T^{-1}M_2T)T^{-1} \\ \Gamma_0T^{-1} \end{bmatrix} = \begin{bmatrix} \Gamma_0T^{-1}\chi_{n-1}(M_2) \\ \vdots \\ \Gamma_0T^{-1}\chi_1(M_2) \\ \Gamma_0T^{-1} \end{bmatrix}$$

Obviously Γ_0 and M_1 generate the same feedback gain as Γ_0T^{-1} and $M_2 = TM_1T^{-1}$. (Note that $\Gamma_0T^{-1} \in \mathfrak{G}(M_2)$ since $\Gamma_0 \in \mathfrak{G}(M_1)$, and, if $\mathfrak{C}\Gamma$ is invertible, so is $\mathfrak{C}\Gamma T^{-1}$.) This shows that any similarity transformation of the matrix M can be used to generate the same class of feedback gain matrices. Thus M can be chosen to have some fixed structure, leaving Γ_0 as the free design parameter. This fixed structure could for instance be the real Jordan form (see Appendix A for definitions of the Jordan forms).
All together this can be summed up in the following lemma:

Lemma 5.1 *Let $M \in \mathfrak{M}_\pi$ and let J_M be a real Jordan form of M. Then $\mathfrak{L}_\Gamma(M) = \mathfrak{L}_\Gamma(J_M)$.*

Proof This follows from the discussion above if we let M_2 be the real Jordan form of M_1. ■

If the desired poles of the closed loop system are distinct, all $M \in \mathfrak{M}_\pi$ (see Definition 3.3) are similar and $\mathfrak{L}_\Gamma(M) = \mathfrak{L}_\Gamma$. However, if there are multiple poles among the desired poles, this is not the case. Diagonalizable matrices will generate a different class of feedback gain matrices than cyclic[1] matrices, and, if the multiplicity of some zero of $p(s)$ is higher than two, the matrix M can be chosen to be neither diagonalizable nor cyclic, and this will generate yet other classes. Some care must be taken in the choice of M in these cases. The conclusion, though, is that as soon as the structure of M is chosen, the matrix Γ_0 can be regarded as the design parameter.
The structure of M is closely related to the structure of the closed loop system. In fact, if $\pi \in \mathfrak{P}_{cop}$, $A - BL$ will have the same Jordan form as M.

[1]See Appendix A for a definition of cyclic matrices.

Lemma 5.2 *Assume that $\pi \in \mathfrak{P}_{cop}$. $\mathfrak{L}_\Gamma(M) \neq 0$ if and only if $M \in \mathfrak{M}_\pi$. Furthermore, if $M \in \mathfrak{M}_\pi$ and $L \in \mathfrak{L}_\Gamma(M)$, then $A - BL$ is similar to M.*

Proof Let $\pi \in \mathfrak{P}_{cop}$. First we show that $A - BL$ and M are similar under the assumption that $\mathfrak{L}_\Gamma(M)$ is nonempty. Then it is shown that $\mathfrak{L}_\Gamma(M)$ is nonempty if $M \in \mathfrak{M}_\pi$.
Assume that $L = \Gamma_0 (\mathfrak{C}\Gamma)^{-1} p(A) \in \mathfrak{L}_\Gamma(M)$. According to Corollary 4.2 $T = p(A)^{-1}\mathfrak{C}\Gamma$ is the unique solution (since $\pi \in \mathfrak{P}_{cop}$) of the Sylvester equation

$$AT - TM = B\Gamma_0 \tag{5.1}$$

Hence, we have

$$L = \Gamma_0 T^{-1} \quad \Longrightarrow \quad T^{-1}(A - BL)T = M$$

Thus, $A - BL$ is similar to M if $L \in \mathfrak{L}_\Gamma(M)$. Furthermore, since $\mathfrak{L}_\Gamma(M) \subseteq \mathfrak{L}_\pi$ (follows from Theorem 3.1) we have that $M \in \mathfrak{M}_\pi$ which proves that $\mathfrak{L}_\Gamma(M)$ is nonempty only if $M \in \mathfrak{M}_\pi$.
Next we show that $\mathfrak{L}_\Gamma(M)$ is nonempty. Let $L \in \mathfrak{L}_\pi$ and $M \in \mathfrak{M}(L) \subseteq \mathfrak{M}_\pi$. That is, L gives the desired poles and M is similar to $A - BL$. By definition there exist a T such that $T^{-1}(A-BL)T = M$. By setting $\Gamma_0 = LT$ the Sylvester equation (5.1) is obtained. Again, according to Corollary 4.2 the unique solution of (5.1) is $T = p(A)^{-1}\mathfrak{C}\Gamma$ (Γ depends on M in accordance with Lemma 3.2). Thus $L = \Gamma_0 (\mathfrak{C}\Gamma)^{-1} p(A)$ belongs to $\mathfrak{L}_\Gamma(M)$. Hence $\mathfrak{L}_\Gamma(M) \neq \emptyset$ for $M \in \mathfrak{M}_\pi$. ∎

The above motivates for the class $\mathfrak{M}_\pi$. The following example highlights the importance of the choice of M.

Example 5.1 Consider the continuous-time system

$$\frac{d}{dt}x = Ax + Bu, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has all poles in the origin. The system is controllable and the controllability matrix is

$$\mathfrak{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose that it is desired to have all poles of the closed loop system in -1, so that the desired characteristic polynomial is

$$p(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1 \quad \Longrightarrow \quad \pi \in \mathfrak{P}_{cop}$$

and that M is chosen as the negative identity matrix, $M = -I$, which clearly has all its eigenvalues in -1. Let

$$\Gamma_0 = \begin{bmatrix} \gamma_1^T \\ \gamma_2^T \end{bmatrix}, \quad \gamma_1, \gamma_2 \in \mathbb{R}^3$$

Then

$$\mathfrak{C}\Gamma = \begin{bmatrix} 2\gamma_1^T \\ \gamma_1^T \\ \gamma_2^T \end{bmatrix}$$

which is singular for every choice of Γ_0. This follows since the first and second rows are linearly dependent. ◦

In Example 5.1 it is concluded that $\mathfrak{G}(-I) = \emptyset$, and $\mathfrak{L}_\Gamma(-I) = \emptyset$. In fact, $\mathfrak{L}(-I) = \emptyset$ (see Definition 3.4), and the reason for this is that $-I \notin \mathfrak{M}_\pi$. This can be seen if we take $M = -I \in \mathfrak{M}_\pi$. With this choice there exists some nonsingular matrix T such that

$$T^{-1}(A - BL)T = -I$$

i.e., such that

$$A - BL = T(-I)T^{-1} = -I$$

All vectors in $\mathbb{R}^3$ would then be eigenvectors of $A - BL = -I$. For instance, e_1^T would be a left eigenvector. However, $e_1^T B = 0$, so with the aid of the PBH eigenvector test (see Theorem B.3 in Appendix B) it can be concluded that the closed loop system is uncontrollable. From this a contradiction arises, because state feedback preserves controllability. If the closed loop system is uncontrollable there exists a vector $q \neq 0$ such that

$$q^T(A - BL) = \lambda q^T, \quad q^T B = 0$$

according to the PBH eigenvector test. This implies that

$$\lambda q^T = q^T(A - BL) = q^T A$$

i.e., the open loop system is uncontrollable as well. This contradicts the fact that the open loop system is controllable in this case. Thus $\mathfrak{L}(-I) = \emptyset$ and $M = -I \notin \mathfrak{M}_\pi$. The above argumentation can be extended for arbitrary systems of arbitrary order. That is, for any scalar α the matrix αI cannot belong to $\mathfrak{M}_\pi$ (at least for systems with $n > m$).
Lemma 5.2 states that M must belong to $\mathfrak{M}_\pi$. Example 5.1 and the discussion above, show that there may exist $M \notin \mathfrak{M}_\pi$ for which $\det(sI - M) = p(s)$. The

question then is how the class $\mathfrak{M}_\pi$ is structured. In Appendix B Rosenbrock's control structure theorem [45] (Theorem B.7) is found, and with the aid of this theorem $\mathfrak{M}_\pi$ can be completely characterized. Rosenbrock's theorem shows how the Smith form of the matrix pencil $sI - A + BL$ is restricted by the controllability indices of the open loop system. The controllability indices are defined in Definition B.3 in Appendix B. The Smith form is discussed at the end of Appendix A (Definition A.8 and Theorem A.5). A discussion about how the Smith form of a matrix pencil $sI - H$ is related to the Jordan form of $H \in \mathbb{R}^{n\times n}$ is also found at the end of Appendix A.
The Jordan form of $A - BL$ is the key issue here. Since every $M \in \mathfrak{M}_\pi$ is similar to $A - BL$ for some $L \in \mathfrak{L}_\pi$, M and $A - BL$ will have the same Jordan form if $L \in \mathfrak{L}_\Gamma(M)$. Thus, any restrictions on the Jordan form of possible closed loop system matrices $A - BL$ are valid for M as well. These restrictions concern the geometric multiplicity (see Appendix A) of the eigenvalues of the closed loop system. That is, the number of Jordan blocks in the Jordan form of $A - BL$ and M that are associated to the same eigenvalue. At most the geometric multiplicity of any eigenvalue can equal the number of inputs. This is what is violated in Example 5.1. There the geometric multiplicity of -1 is three, while there only are two inputs.
The Matlab function for pole assignment, **place**, is based on an algorithm by Kautsky, Nichols and van Dooren [26], which implies that $A - BL$ is diagonalizable. Since the geometric and algebraic multiplicities are equal for diagonalizable matrices, this algorithm is not applicable in Example 5.1, nor in any case where a desired pole has higher multplicity than the number of inputs. This is also notified in the help function for the **place** function. The exact presumptions for the algorithm to work are described in [26]. More details on how the Jordan form of $A - BL$ and M is restricted is found in Theorem B.7 in Appendix B.

5.2 Total pole assignment

In this section the special case $\pi \in \mathfrak{P}_{cop}$ ($p(s)$ and $a(s)$ are coprime) is considered. Of particular importance is the fact that $p(A)$ is nonsingular when $\pi \in \mathfrak{P}_{cop}$. This makes it possible to utilize the similarity transformation interpretation presented in Chapter 4.
As mentioned above, the parameterization presented in Chapter 3 is useful if it belongs to a class of feedback gains that is sufficiently large. When $\pi \in \mathfrak{P}_{cop}$ the parameterization is very successful since all possible feedback gain matrices can be parameterized in this way. We have

Theorem 5.3 *Let (A, B) be a controllable pair. Then $\mathfrak{L}_\Gamma = \mathfrak{L}_\pi$ if $\pi \in \mathfrak{P}_{cop}$.*

An aid in the proof of this theorem, but also a strong and useful result in itself, is this lemma:

Lemma 5.4 *Let (A, B) be a controllable pair and let $M \in \mathfrak{M}_\pi$. Then*

$$\mathfrak{L}_\Gamma(M) = \mathfrak{L}(M)$$

if $\pi \in \mathfrak{P}_{cop}$.

Proof We first show that $\mathfrak{L}_\Gamma(M) \subseteq \mathfrak{L}(M)$. Assume that $L \in \mathfrak{L}_\Gamma(M)$ and that $\pi \in \mathfrak{P}_{cop}$. Then $\mathfrak{C}\Gamma$ is nonsingular by assumption. According to Corollary 4.2

$$\Omega = (p(A))^{-1}\mathfrak{C}\Gamma$$

is the unique solution to the Sylvester equation $A\Omega - \Omega M = B\Gamma_0$. Since

$$L \in \mathfrak{L}_\Gamma(M) \quad \Longrightarrow \quad L = \Gamma_0(\mathfrak{C}\Gamma)^{-1}p(A) = \Gamma_0\Omega^{-1}$$

it follows that

$$A\Omega - \Omega M = BL\Omega \quad \Longleftrightarrow \quad \Omega^{-1}(A - BL)\Omega = M$$

and hence that $L \in \mathfrak{L}(M)$. Since this is valid for any $L \in \mathfrak{L}_\Gamma(M)$, $\mathfrak{L}_\Gamma(M) \subseteq \mathfrak{L}(M)$ must hold.
Next we prove that $\mathfrak{L}(M) \subseteq \mathfrak{L}_\Gamma(M)$. Assume that $L \in \mathfrak{L}(M)$. Then there exists at least one nonsingular matrix T such that $T^{-1}(A - BL)T = M$. If we set $\Gamma_0 = LT$ the Sylvester equation $AT - TM = B\Gamma_0$ is obtained. According to Corollary 4.2 this Sylvester equation has the unique solution $T = (p(A))^{-1}\mathfrak{C}\Gamma$. Thus

$$L = \Gamma_0 T^{-1} = \Gamma_0(\mathfrak{C}\Gamma)^{-1}p(A) \in \mathfrak{L}_\Gamma(M)$$

This holds for any $L \in \mathfrak{L}(M)$, and thus $\mathfrak{L}(M) \subseteq \mathfrak{L}_\Gamma(M)$. ■

Theorem 5.3 can now be proven.

Proof of Theorem 5.3 That $\mathfrak{L}_\Gamma \subseteq \mathfrak{L}_\pi$ follow from Theorem 3.1. Assume now that $\pi \in \mathfrak{P}_{cop}$. Pick $L \in \mathfrak{L}_\pi$ arbitrarily. Obviously $L \in \mathfrak{L}(A - BL)$. Then, according to Lemma 5.4, $L \in \mathfrak{L}_\Gamma(A - BL) \subseteq \mathfrak{L}_\Gamma$. Since this holds for any $L \in \mathfrak{L}_\pi$, $\mathfrak{L}_\pi \subseteq \mathfrak{L}_\Gamma$ must hold. Thus $\mathfrak{L}_\Gamma = \mathfrak{L}_\pi$ if $\pi \in \mathfrak{P}_{cop}$. ■

Theorem 5.3 states that, in the total pole assignment problem, every possible feedback gain can be parameterized as in Theorem 3.1 and Lemma 3.2. It is stressed that this result is only valid for $\pi \in \mathfrak{P}_{cop}$. For the partial pole assignment problem the situation is less favourable. The similarity transformation interpretation makes the analysis of $\mathfrak{L}_\Gamma$ rather simple for $\pi \in \mathfrak{P}_{cop}$, but this interpretation is not applicable for $\pi \notin \mathfrak{P}_{cop}$. This makes it more difficult to

characterize the class $\mathfrak{L}_\Gamma$ for $\pi \notin \mathfrak{P}_{cop}$. However, it can be shown that not all feedback gains can be generated with the parameterization for partial pole assignment. In Section 5.3 the case $\pi \notin \mathfrak{P}_{cop}$ is further discussed.
As mentioned in Section 5.1 the matrix M is regarded as constant, while Γ_0 is free. This is the situation in Lemma 5.4, but not in Theorem 5.3 where M is regarded as a free parameter as well (confined to $\mathfrak{M}_\pi$). As soon as M has been chosen only Lemma 5.4 is applicable. When all desired poles of the closed loop system are distinct the choice of M is of less importance, since all admissable M are similar in that case. This means that $\mathfrak{L}_\Gamma(M) = \mathfrak{L}_\Gamma = \mathfrak{L}_\pi$. However, if some of the desired poles are multiple the choice of M is crucial. According to the Lemmas 5.1 and 5.2 the closed loop system matrix $A - BL$ and M have the same Jordan form if $L \in \mathfrak{L}_\Gamma(M)$. For multiple poles there are multiple possible Jordan forms. Assume that $M_1, M_2 \in \mathfrak{M}_\pi$ have different Jordan forms. Then $\mathfrak{L}_\Gamma(M_1)$ and $\mathfrak{L}_\Gamma(M_2)$ are disjoint, since M_1 and M_2 are not similar. An interpretation of this is that the design is separated into two steps. In the first step the Jordan form of the closed loop system is determined by the choice of M. In the second step Γ_0 is used to generate a feedback gain, and regardless of how Γ_0 is chosen the closed loop system will have the desired Jordan form.
Below some examples that illustrate these results are given.

Example 5.2 Reconsider the system in Example 3.1, *i.e.*, the continuous-time system

$$\frac{d}{dt}x = Ax + Bu, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with the desired characteristic polynomial $p(s) = s^3 + 3s^2 + 3s + 1$ (so that $\pi \in \mathfrak{P}_{cop}$) and

$$M = K_\pi = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$

The feedback gain matrix obtained in Example 3.1 yields a closed loop system matrix $A - BL$ that is reminiscent of a companion matrix. If the feedback gain matrix

$$L' = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 3 & 3 \end{bmatrix}$$

is used instead of L in Example 3.1, the closed loop system matrix

$$A - BL' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$$

is obtained. Indeed, this is the lower companion matrix associated to $p(s)$. Since all companion matrices associated to the same polynomial are similar, L' belongs to $\mathfrak{L}_\Gamma(K_\pi)$ according to Lemma 5.4. Thus there is a Γ_0 that would yield L' if used in Equation (3.21). Hence

$$L' = \Gamma_0(\mathfrak{C}\Gamma)^{-1}p(A) \iff \Gamma_0 = L'\left(p(A)\right)^{-1}\mathfrak{C}\Gamma$$

must be satisfied. Recall that the matrix Γ depends on Γ_0, as is explained in Lemma 3.2. With

$$\Gamma_0 = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix}$$

the equation

$$L'\left(p(A)\right)^{-1}\mathfrak{C}\Gamma = \\ = \begin{bmatrix} -3g_{21} - 3g_{22} - g_{23} & g_{21} & g_{22} \\ 3g_{11} + g_{12} + 9g_{21} + 9g_{22} + 3g_{23} & 3g_{12} + g_{13} + 3g_{21} & -g_{11} - 3g_{12} - 3g_{22} \end{bmatrix} = \\ = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix}$$

is obtained. The solution is

$$\begin{cases} g_{12} & = g_{21} = 0 \\ g_{13} & = g_{22} \\ g_{23} & = -g_{11} - 3g_{13} \end{cases} \quad \text{or} \quad \Gamma_0 = \begin{bmatrix} g_{11} & 0 & g_{13} \\ 0 & g_{13} & -g_{11} - 3g_{13} \end{bmatrix}$$

where g_{11} and g_{13} can be chosen arbitrarily. There is an additional restriction in the choice of g_{11} and g_{13}; Γ_0 must belong to $\mathfrak{G}(M)$ so that $\mathfrak{C}\Gamma$ is nonsingular. This means that the determinant of $\mathfrak{C}\Gamma$ must be nonzero. Here $\det \mathfrak{C}\Gamma = (g_{11} + g_{13})^3$. Hence, as long as $g_{11} \neq -g_{13}$ any choice of g_{11} and g_{13} in Γ_0 above will give the feedback gain matrix L'. ○

Example 5.3 In Example 5.1 it is shown that $\mathfrak{L}(-I) = \emptyset$ for the system

$$\frac{d}{dt}x = Ax + Bu, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with the desired characteristic polynomial $p(s) = s^3+3s^2+3s+1$, since $-I \notin \mathfrak{M}_\pi$. Another choice of M, that belongs to $\mathfrak{M}_\pi$, is

$$M = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that M is not cyclic and thus not similar to any companion matrix. Hence, the feedback gain in Example 5.2 cannot be obtained with this M, since

$$\mathfrak{L}(M) \cap \mathfrak{L}(K_\pi) = \emptyset$$

In order to evaluate $\mathcal{C}\Gamma$ we notice that

$$\chi_1(M) = M + 3I = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \chi_2(M) = M^2 + 3M + 3I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With

$$\Gamma_0 = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix}$$

the matrix $\mathcal{C}\Gamma$ becomes

$$\mathcal{C}\Gamma = \begin{bmatrix} 2g_{11} & g_{11} + 2g_{12} & 2g_{13} \\ g_{11} & g_{11} + g_{12} & g_{13} \\ g_{21} & g_{21} + g_{22} & g_{23} \end{bmatrix}$$

If $g_{11} = 0$ the matrix is singular, so $g_{11} \neq 0$ is a necessary condition for $\mathcal{C}\Gamma$ to be nonsingular. Choosing

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

yields

$$\mathcal{C}\Gamma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \iff (\mathcal{C}\Gamma)^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$L = \Gamma_0 \, (\mathcal{C}\Gamma)^{-1} p(A) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies A - BL = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The minimal polynomial (see Definition A.4 in Appendix A) of M is $(s+1)^2$, *i.e.*, $(M + I)^2 = 0$. According to Lemma 5.2 M and $A - BL$ are similar, and therefore they have the same minimal polynomial. Indeed, $(A - BL + I)^2 = 0$. $\circ$

5.3 Partial pole assignment

As have been discussed in Section 5.2 above the property $\pi \in \mathfrak{P}_{cop}$ implies that the parameterized gains are all gains; $\mathfrak{L}_\Gamma = \mathfrak{L}_\pi$.
When the desired poles of the closed loop system include some of the open loop system poles, *i.e.*, $\pi \notin \mathfrak{P}_{cop}$, the situation is more complex. It is obvious that $\mathfrak{L}_\Gamma \subseteq \mathfrak{L}_\pi$, and it easy to construct examples for which the parameterization cannot be used to generate all gains. For instance, if $p(s) = a(s)$ (all the poles should be unchanged), then

$$L = \Gamma_0(\mathfrak{C}\Gamma)^{-1}p(A) = 0$$

since $a(A) = 0$ according to Caley-Hamilton's theorem. Thus $L = 0$ is the only gain in $\mathfrak{L}_\Gamma$ for this case. A simple example shows that nonzero gains can be found.

Example 5.4 For the system

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

which has all poles in the origin, the feedback gain

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

yields the closed loop system

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x$$

which still has all poles in the origin. Thus, there exist $L \neq 0$ that leaves the poles unchanged. ∘

At the end of this section an algorithm (by Moore 1976 [40]) is outlined. This algorithm can be used to construct $L \notin \mathfrak{L}_\Gamma$ in the general case for $\pi \notin \mathfrak{P}_{cop}$. The conclusion is that the parameterization is less successful for partial pole assignment if compared to the situation for total pole assignment. This is further established in this section and in Section 6.3.
Still, the parameterization applies in many cases for $\pi \notin \mathfrak{P}_{cop}$. That is, it can be used to generate feedback gains, as is illustrated in the following example.

Example 5.5 Consider the continuous-time system

$$\frac{d}{dt}x = Ax + Bu, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has the controllability matrix

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This system has all poles in the origin. Suppose that the desired characteristic polynomial is $p(s) = s^3 + 2s^2 + s$, and that

$$M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The pole in the origin is chosen merely to keep the computational burden down. Note that this pole appears both in the open loop and closed loop system, and hence that $\pi \notin \mathfrak{P}_{cop}$. With

$$\Gamma_0 = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix}$$

we obtain

$$\mathcal{C}\Gamma = \begin{bmatrix} g_{11} + g_{21} & g_{12} + g_{22} & 2g_{13} + g_{23} \\ g_{21} & g_{22} & g_{13} + 2g_{23} \\ 0 & 0 & g_{23} \end{bmatrix}$$

Now Γ_0 should be chosen so that $\mathcal{C}\Gamma$ is nonsingular. It is readily seen that the determinant is

$$\det \mathcal{C}\Gamma = g_{23}(g_{11}g_{22} - g_{12}g_{21})$$

With

$$\Gamma_0 = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

we have

$$\mathcal{C}\Gamma = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Longleftrightarrow \quad (\mathcal{C}\Gamma)^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Furthermore

$$p(A) = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying Equation (3.21) gives the feeback gain matrix

$$L = \Gamma_0 \left(\mathfrak{C}\Gamma\right)^{-1} p(A) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and the closed loop system matrix

$$A - BL = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\circ$

In Example 5.4 it is shown that the parameterization fails in the case $p(s) = a(s)$, since then $p(A) = 0$. This is of course the most extreme case of partial pole assignment. However, the matrix $p(A)$ restricts the parameterization for all $\pi \notin \mathfrak{P}_{cop}$, because it is singular. This is readily seen; let $q \neq 0$ be a right eigenvector of A that corresponds to the eigenvalue λ. Then

$$p(A)q = p(\lambda)q$$

If λ, which is an open loop system pole, is a zero of $p(s)$ we have $p(A)q = 0$. Thus $p(A)$ is singular for $\pi \notin \mathfrak{P}_{cop}$. A consequence of this is stated in this lemma:

Lemma 5.5 *If $Aq = \lambda q$ and $p(\lambda) = 0$, then $(A - BL)q = \lambda q$ for $L \in \mathfrak{L}_\Gamma$.*

Proof According to Theorem 3.1 $L = \Gamma_0 \left(\mathfrak{C}\Gamma\right)^{-1} p(A)$. If $Aq = \lambda q$ and $p(\lambda) = 0$, then $p(A)q = 0$ and

$$(A - BL)q = Aq - B\Gamma_0 \left(\mathfrak{C}\Gamma\right)^{-1} p(A)q = \lambda q$$

$\blacksquare$

This means that eigenvectors, corresponding to common eigenvalues (poles) of the open and closed loop system, are invariant when feedback gain matrices in $\mathfrak{L}_\Gamma$ are used. In other words, it is impossible to modify the eigenvectors, that correspond to unchanged eigenvalues, with $L \in \mathfrak{L}_\Gamma$. This is the situation for single-input systems, for which the unique feedback gain have no effect on eigenvectors that correspond to unchanged eigenvalues. For multi-input systems the additional degree of freedom can be used in order to modify the eigenvectors, and this of course includes eigenvectors to unchanged eigenvalues. This will first be illustrated in a simple example, and then for a more general case by the use of the argumentation in Moore 1976 [40].

Example 5.6 In Example 5.5 $q = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$ is the eigenvector of A that corresponds to the eigenvalue in the origin, *i.e.*, $Aq = 0$. Indeed, $(A - BL)q = 0$ as well, and this is the only possibility for $L \in \mathfrak{L}_\Gamma$ according to Lemma 5.5. If we use

$$L' = \begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

instead, the closed loop system matrix

$$A - BL' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

is obtained. Again the closed loop system has one pole in the origin and a double pole in -1, but here $q' = \begin{bmatrix}1 & 0 & 1\end{bmatrix}^T$ is the eigenvector corresponding to zero, *i.e.*, $(A - BL')q' = 0$. ○

In Moore 1976 [40] it is shown how state feedback gains can be constructed in order to achieve desired eigenvalues and eigenvectors for the closed loop system matrix $A - BL$. In [40] distinct eigenvalues in $\mathbb{C}$ are considered, while the more general case with generalized eigenvectors is treated in Klein and Moore 1977 [31]. Below an outline of an algorithm for eigenstructure assignment, based on Proposition 1 in [40], is given. For simplicity only real eigenvalues are considered. For the controllable pair (A, B), let

$$S_\lambda = \begin{bmatrix}\lambda I - A & B\end{bmatrix} \tag{5.2}$$

The columns of the compatibly partitioned matrix

$$K_\lambda = \begin{bmatrix} N_\lambda \\ M_\lambda \end{bmatrix} \tag{5.3}$$

constitute a basis for the null space of S_λ. Thus $S_\lambda K_\lambda = 0$. It can be shown that N_λ have full (column) rank if B have full rank.
Assume that $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ are the desired poles of the closed loop system. Let $q_1, \dots, q_n \in \mathbb{R}^n$ be linearly independent vectors for which

$$q_i \in \operatorname{span}\{N_{\lambda_i}\} \quad \text{for} \quad i = 1, \dots, n \tag{5.4}$$

We will now construct an L such that $A - BL$ will have $\lambda_1, \dots, \lambda_n$ and $q_1, \dots, q_n$ as eigenvalues and eigenvectors respectively. The condition (5.4) can be expressed as $q_i = N_{\lambda_i} k_i$ for some real vector k_i. We have

$$0 = S_{\lambda_i} K_{\lambda_i} k_i = (\lambda_i I - A) q_i + B M_{\lambda_i} k_i$$

If we choose a feedback gain so that

$$L q_i = M_{\lambda_i} k_i, \quad i = 1, \dots, n \tag{5.5}$$

then

$$(\lambda_i I - A + BL) q_i = 0$$

The existence of this L is guaranteed since $q_1, \dots, q_n$ are linearly independent. To get a closed form expression for this L, let

$$Q = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}, \quad W = \begin{bmatrix} M_{\lambda_1} k_1 & \dots & M_{\lambda_n} k_n \end{bmatrix}$$

The feedback gain $L = WQ^{-1}$ then satifies (5.5) for $i = 1, \dots, n$.
In [40] and [31] it is shown that this argumentation, with minor modifications, applies to the general case.
Of course some of the desired poles may be open loop poles as well. An eigenvector corresponding to such an eigenvalue, say λ_j, can be chosen arbitrarily in the subspace span$\{N_{\lambda_j}\}$. As B is assumed to have full rank, the dimension of this subspace is at least m, the number of inputs. Thus, the eigenvector of $A - BL$ need not be equal to the eigenvector of A for λ_j. Hence there exist feedback gains $L \in \mathfrak{L}_\pi$ that cannot be reached with the parameterization for the case $\pi \notin \mathfrak{P}_{cop}$.
For some cases the parameterization is not applicable at all ($\mathfrak{G}(M) = \emptyset$ for all $M \in \mathfrak{M}_\pi$). One approach, further discussed in Cavin and Bhattacharyya [9], to avoid these problems is to perform the pole assignment in two steps so that the poles are modified in each step.

5.4 Redundancy in the design parameters

Example 5.2 shows that the same feedback gain matrix L can be generated from a whole class of matrices Γ_0. This fact is true in the general case as well. To see this, let $L = \Gamma_0(\mathcal{C}\Gamma)^{-1} p(A)$ and $L' = \Gamma_0'(\mathcal{C}\Gamma')^{-1} p(A)$, where we relate Γ_0' and Γ_0 to each other as $\Gamma_0' = \Gamma_0 \varphi(M)$. $\varphi(\cdot)$ is an arbitrary polynomial such that $\varphi(M)$ is nonsingular. We then have

$$\Gamma_0'(\mathcal{C}\Gamma')^{-1} = \Gamma_0 \varphi(M)(\mathcal{C}\Gamma')^{-1} = \Gamma_0(\mathcal{C}\Gamma'(\varphi(M))^{-1})^{-1} \tag{5.6}$$

and

$$\Gamma'\varphi(M)^{-1} = \begin{bmatrix} \Gamma_0\varphi(M)\chi_{n-1}(M)\varphi(M)^{-1} \\ \vdots \\ \Gamma_0\varphi(M)\chi_1(M)\varphi(M)^{-1} \\ \Gamma_0\varphi(M)\varphi(M)^{-1} \end{bmatrix} = \begin{bmatrix} \Gamma_0\chi_{n-1}(M) \\ \vdots \\ \Gamma_0\chi_1(M) \\ \Gamma_0 \end{bmatrix} = \Gamma \tag{5.7}$$

Thus $L' = L$.
This illustrates the redundancy in the parameterization. An intuitive reasoning is that L has nm entries which are to be chosen so that the closed loop system gets poles that coincide with n predefined complex numbers (the desired poles). This should leave $(n-1)m$ degrees of freedom in the choice of L, while all nm entries of Γ_0 can be chosen arbitrarily (as long as $\Gamma_0 \in \mathfrak{G}(M)$). This indicates a certain degree of "overparameterization". In the case when $\pi \in \mathfrak{P}_{cop}$ the class of Γ_0 that for a given M generates identical feedback gain matrices can be parameterized in a neat way, as is shown in the following theorem.

Theorem 5.6 *Let $\pi \in \mathfrak{P}_{cop}$ and let $L_1, L_2 \in \mathfrak{L}_\Gamma(M)$ be such that*

$$L_1 = \Gamma_0^1(\mathfrak{C}\Gamma^1)^{-1}p(A) \quad \textit{and} \quad L_2 = \Gamma_0^2(\mathfrak{C}\Gamma^2)^{-1}p(A)$$

for some $\Gamma_0^1, \Gamma_0^2 \in \mathfrak{G}(M)$. Then $L_1 = L_2$ if and only if $\Gamma_0^2 = \Gamma_0^1\phi$ where ϕ is nonsingular and commutes with $M \in \mathfrak{M}_\pi$, i.e., *$\phi M = M\phi$.*

Proof First the sufficiency is shown. Let $L_1 = \Gamma_0^1(\mathfrak{C}\Gamma^1)^{-1}p(A)$, and let ϕ, nonsingular, commute with M. Then the same calculations as in (5.6) and (5.7) show that

$$L_2 = \Gamma_0^2(\mathfrak{C}\Gamma^2)^{-1}p(A) = L_1$$

if $\Gamma_0^2 = \Gamma_0^1\phi$.
To show the necessity, let $L^* \in \mathfrak{L}_\Gamma(M)$. In Chapter 4 it is shown how the feedback gain can be obtained by solving a Sylvester equation, and how this can be interpreted in terms of similarity transformations. Corollary 4.2 states that $(p(A))^{-1}\,\mathfrak{C}\Gamma$ is the unique solution to the Sylvester equation

$$A\Omega - \Omega M = B\Gamma_0$$

when $\pi \in \mathfrak{P}_{cop}$. Thus the feedback gain can be obtained as $L = \Gamma_0\Omega^{-1}$ whenever Ω is nonsingular. Define the class

$$\mathfrak{T}(L^*, M) \triangleq \{T \in \mathbb{R}^{n\times n} | T^{-1}(A - BL^*)T = M\}$$

for an arbitrary and fixed $L^* \in \mathfrak{L}_\Gamma(M)$. Thus, for any $T \in \mathfrak{T}(L^*, M)$ the choice $\Gamma_0 = L^*T$ generates L^*. This Γ_0 belongs to $\mathfrak{G}(M)$ since T is nonsingular.

Furthermore, *every* Γ_0 that together with M generates exactly L^* is given by $\Gamma_0 = L^*T$ where $T \in \mathfrak{T}(L^*, M)$.

Take $T_1, T_2 \in \mathfrak{T}(L^*, M)$ and let $\Gamma_0^1 = L^*T_1$ and $\Gamma_0^2 = L^*T_2$. By definition T_1 and T_2 are nonsingular. We have

$$\Gamma_0^2 = L^*T_2 = L^*T_1(T_1)^{-1}T_2 = \Gamma_0^1(T_1)^{-1}T_2$$

Thus $\Gamma_0^2 = \Gamma_0^1\phi$ where $\phi = (T_1)^{-1}T_2$ is nonsingular. Also,

$$\phi M = (T_1)^{-1}T_2M = (T_1)^{-1}(A - BL^*)T_2 = M(T_1)^{-1}T_2 = M\phi$$

Hence $\phi = (T_1)^{-1}T_2$ commutes with M. ■

Remark Any polynomial in M, like $\varphi(M)$ in (5.6) and (5.7), commutes with M. Furthermore, if M is cyclic it is only polynomials in M that commute with M (see for instance Chapter 3.2.4 in Horn and Johnson [23]). According to Caley-Hamilton's theorem $\varphi(\cdot)$ need not be of higher degree than $n-1$. Also, $\varphi(M)$ is nonsingular if and only if $\varphi(s)$ and $p(s)$ are coprime. This provides some insight into the intuitive discussion above; the "excess" of n degrees of freedom in Γ_0 can be interpreted as the n coefficients in a polynomial $\varphi(\cdot)$ of degree $n-1$.

Theorem 5.6 reminds of Lemma 5.1 in Section 5.1. In particular the calculations leading to the results are similar. The difference is that in Lemma 5.1 both M and Γ_0 are free parameters, while in Theorem 5.6 M is constant and Γ_0 is the only free parameter.

The calculations in (5.6) and (5.7) are of course valid for the case when $\pi \notin \mathfrak{P}_{cop}$ as well. Thus, if ϕ is nonsingular and commutes with M, then $\Gamma_0^1 \in \mathfrak{G}(M)$ and $\Gamma_0^2 = \Gamma_0^1\phi$ will generate identical feedback gain matrices L. However, it is an open problem whether the whole class of Γ_0 generating the same L can be parameterized in this way when $\pi \notin \mathfrak{P}_{cop}$.

Chapter 6

On the existence of the parameterized feedback gains

The parameterization of feedback gain matrices for pole assignment presented in Theorem 3.1 and Lemma 3.2 relies on the existence of the inverse of the matrix $\mathcal{C}\Gamma$, where $\mathcal{C}$ is the controllability matrix for the system (3.1). The matrix Γ, given by Equations (3.17) and (3.18), depends on the matrices M and Γ_0. In this chapter the role of Γ_0 and M with respect to the nonsingularity of $\mathcal{C}\Gamma$ is considered. In Section 6.1 some necessary conditions for the nonsingularity of $\mathcal{C}\Gamma$ will be given. In Section 6.2 it is shown that the existence of the inverse of $\mathcal{C}\Gamma$ is almost guaranteed for the total pole assignment problem. To show this the concept of genericity is adopted. It is shown in Section 6.3 that for certain open loop systems it is impossible to find Γ_0 and M so that $\mathcal{C}\Gamma$ has full rank in the partial pole assignment case. The matrix $p(A)$ is singular when $\pi \notin \mathfrak{P}_{cop}$, and this is the reason for the shortcomings of the parameterization in this case. A discussion on the role of $p(A)$ in the parameterization is given in Section 6.4.

6.1 General necessary conditions

For $\mathcal{C}\Gamma$ to be invertible it must have rank n. There are several ways to write the product $\mathcal{C}\Gamma$, and below we will elaborate on this in order to give necessary conditions for nonsingularity. A well known property (see for instance [23]) is that $\text{rank}\mathcal{C}\Gamma \leq \min\{\text{rank}\mathcal{C}, \text{rank}\Gamma\}$. Thus, obvious necessary conditions for nonsingularity of $\mathcal{C}\Gamma$ is that $\text{rank}\mathcal{C} = n$ and $\text{rank}\Gamma = n$. Due to the controllability assumption $\text{rank}\mathcal{C} = n$. However, some basic manipulations of the product $\mathcal{C}\Gamma$ reveals stronger conditions, which offer a nice interpretation in terms of linear system theory.

Define the matrix

$$\Pi \triangleq \begin{bmatrix} p_{n-1} & \cdots & p_1 & 1 \\ \vdots & \iddots & 1 & 0 \\ p_1 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \tag{6.1}$$

i.e., a Hankel matrix in the coefficients of $p(\cdot)$. Then Γ can be factorized into this Hankel matrix and the observability matrix for the pair (Γ_0, M)

$$\Gamma = \left[\Pi \otimes I_m\right] \begin{bmatrix} \Gamma_0 \\ \Gamma_0 M \\ \vdots \\ \Gamma_0 M^{n-1} \end{bmatrix}$$

The matrix $\Pi \otimes I_m$ is square and has full rank so Γ will have full rank if and only if the observability matrix has full rank.
Now it is possible to write the product $\mathcal{C}\Gamma$ as

$$\begin{aligned} \mathcal{C}\Gamma = \mathcal{C}\left[\Pi \otimes I_m\right] \begin{bmatrix} \Gamma_0 \\ \Gamma_0 M \\ \vdots \\ \Gamma_0 M^{n-1} \end{bmatrix} &= \\ \begin{bmatrix} B\Gamma_0 & AB\Gamma_0 & \dots & A^{n-1}B\Gamma_0 \end{bmatrix} \left[\Pi \otimes I_n\right] \begin{bmatrix} I \\ M \\ \vdots \\ M^{n-1} \end{bmatrix} &= \\ \begin{bmatrix} I & A & \dots & A^{n-1} \end{bmatrix} \left[\Pi \otimes I_n\right] \begin{bmatrix} B\Gamma_0 \\ B\Gamma_0 M \\ \vdots \\ B\Gamma_0 M^{n-1} \end{bmatrix} & \end{aligned} \tag{6.2}$$

The matrices $\begin{bmatrix} I & A & \dots & A^{n-1} \end{bmatrix}$ and $\begin{bmatrix} I & M^T & \dots & (M^{n-1})^T \end{bmatrix}^T$ both have full rank. This expression shows that necessary conditions for the nonsingularity of $\mathcal{C}\Gamma$ are that $(A, B\Gamma_0)$ is controllable and that $(B\Gamma_0, M)$ is observable. Since B is assumed to have full rank, the latter condition is equivalent to that (Γ_0, M) is observable. To see this, apply the PBH eigenvector test (Theorem B.4 in Appendix B) to (Γ_0, M) and $(B\Gamma_0, M)$. If (Γ_0, M) is unobservable there exists an eigenvector q of M such that $\Gamma_0 q = 0$. Then of course $B\Gamma_0 q = 0$ as well, so $(B\Gamma_0, M)$ is also unobservable. On the other hand, if $\Gamma_0 q \neq 0$ for all eigenvectors, then $B\Gamma_0 q \neq 0$, since B has full rank, *i.e.*, the columns are linearly independent.

In the same way $(A, B\Gamma_0)$ is controllable if and only if (A, B) is controllable when Γ_0 have full rank. However, there are no restrictions on the rank of Γ_0. In fact, by choosing Γ_0 as a rank one matrix, *i.e.*, $\Gamma_0 = q\gamma_0^T$ for some vectors q and γ_0^T, it is sufficient to have (A, Bq) controllable and (γ_0^T, M) observable for $\mathcal{C}\Gamma$ to be nonsingular. This of course reduces the system to a single-input system, and it is only applicable when the system matrix A is cyclic.
The discussion so far leads to the following lemma:

Lemma 6.1 *The matrix* $\mathcal{C}\Gamma$ *is nonsingular only if* $(A, B\Gamma_0)$ *is controllable and* (Γ_0, M) *is observable.*

Proof The result follow from (6.2) and the discussion above. An alternative way to show the result is to write the product $\mathcal{C}\Gamma$ like

$$\mathcal{C}\Gamma = \sum_{k=1}^{n} A^{k-1} B\Gamma_0 \chi_{n-k}(M) \tag{6.3}$$

Assume that $(A, B\Gamma_0)$ is uncontrollable. Then, according to the PBH eigenvector test, there exists a left eigenvector $q^T \neq 0$ of A, corresponding to λ, such that $q^T B\Gamma_0 = 0$. Multiplying $\mathcal{C}\Gamma$ with q^T yields

$$q^T \mathcal{C}\Gamma = \sum_{k=1}^{n} q^T A^{k-1} B\Gamma_0 \chi_{n-k}(M) = \sum_{k=1}^{n} \lambda^{k-1} q^T B\Gamma_0 \chi_{n-k}(M) = 0$$

Thus $\mathcal{C}\Gamma$ is singular if $(A, B\Gamma_0)$ is uncontrollable, which proves the necessity of controllability. Similarily, assume that (Γ_0, M) is unobservable. Then, according to the PBH eigenvector test, there exists a right eigenvector $q \neq 0$ of M, corresponding to λ, such that $\Gamma_0 q = 0$. Multiplying $\mathcal{C}\Gamma$ with q yields

$$\mathcal{C}\Gamma q = \sum_{k=1}^{n} A^{k-1} B\Gamma_0 \chi_{n-k}(M) q = \sum_{k=1}^{n} \chi_{n-k}(\lambda) A^{k-1} B\Gamma_0 q = 0$$

Thus $\mathcal{C}\Gamma$ is singular if (Γ_0, M) is unobservable, which proves the necessity of observability. ■

In Chapter 4 it is shown how the feedback gain matrix can be obtained by solving a Sylvester equation. It is also shown that $\mathcal{C}\Gamma$ is a solution to a Sylvester equation. For the case $\pi \in \mathfrak{P}_{cop}$ there are necessary conditions for the nonsingularity of the solution to the Sylvester equation presented in Hearon [21] and in de Souza and Bhattacharyya [14]. These conditions are perfectly in line with the conditions in Lemma 6.1. The difference is that Lemma 6.1 is valid for all π, *i.e.*, for both total and partial pole assignment.
In Example 5.1 it is shown that $\mathfrak{L}_\Gamma(M) = \emptyset$ for $M = -I$. In the light of Lemma 6.1 it is clear that $M = -I$ (or any scalar matrix) is a nonapplicable choice for

M for any system in any situation, because $(\Gamma_0, -I)$ is unobservable for every Γ_0. At least for systems with less input signals than states $(m < n)$ there will always exist some vector q such that $\Gamma_0 q = 0$. Since $-Iq = -q$, $(\Gamma_0, -I)$ is unobservable according to the PBH eigenvector test.
The conditions in Lemma 6.1 are necessary, but not sufficient as is illustrated in the following example.

Example 6.1 Let the system $\frac{d}{dt}x = Ax + Bu$ have the system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e., the open loop system has all poles in the origin. Suppose that $p(s) = s^3 + 5s^2 + 4s + 2$ is the desired characteristic polynomial for the closed system, *i.e.*, a double pole in -1 and one pole in -2. With

$$M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \Gamma_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

both of the necessary conditions are satisfied, *i.e.*, $(A, B\Gamma_0)$ is controllable and (Γ_0, M) is observable. Still

$$\mathfrak{C}\Gamma = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

which is singular. With the same M, but with

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

one gets

$$\mathfrak{C}\Gamma = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

which is nonsingular. ∘

6.2 Existence for total pole assignment

For the case $\pi \in \mathfrak{P}_{cop}$ the matrix $\mathfrak{C}\Gamma$ is nonsingular for almost every $\Gamma_0 \in \mathbb{R}^{m \times n}$. To show this, the concept of genericity, introduced in Wonham 1974 [54], will be

used. This concept has also been adopted in de Souza and Bhattacharyya 1981 [14] for examining the generic rank of the solution to the Sylvester equation. It will be used in a similar manner here. For this reason a short description of the concept of genericity is given.
Consider a mathematical object that depends on a number of parameters. Assume that the object has a property for certain values of these parameters. The object can either have or not have the property, depending on the parameters. It is of interest to be able to decide whether this property is typical, atypical or neither of these. If the property is typical for the object, the property is said to be generic. In other words, this property is generic if the object has the property for almost all values of the parameters.
Consider the N-tuple

$$g = (\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$$

where $\gamma_1, \ldots, \gamma_N$ represent the parameters.

Definition 6.1 A *variety* $V \subset \mathbb{R}^N$ is the set of common zeros of a finite number of polynomials $\varphi_1, \ldots, \varphi_k$ of finite degree, with coefficients in $\mathbb{R}$, *i.e.*,

$$V = \{g \in \mathbb{R}^N | \varphi_i(\gamma_1, \ldots, \gamma_N) = 0, i = 1, \ldots, k\}$$

The variety is *proper* if $V \neq \mathbb{R}^N$, and *nontrivial* if $V \neq \emptyset$. $\circ$

Definition 6.2 A property is *generic* if it fails to hold only for $g \in V$, where V is a nontrivial proper variety. $\circ$

Notice that there must exist parameter values for which the object does not have the property, otherwise the property is not generic.
Suppose that $g_0 \in V$. Then the mathematical object does not have the property in question for g_0. However, every neighborhood of g_0 contains points that do not belong to the variety. This follows from continuity reasons since V is defined as the zeros of a number of polynomials. That is, since $V \neq \mathbb{R}^N$ at least one of the polynomials is not identically zero, and this polynomial will be nonzero for some g in the neighborhood of g_0. In other words, if the property fails to hold for g_0, it can be made to hold for a suitable and arbitrarily small perturbation of g_0.
In this case the mathematical object is $\mathfrak{C}\Gamma$. The property of interest is the nonsingularity of $\mathfrak{C}\Gamma$.

Theorem 6.2 *If $\pi \in \mathfrak{P}_{cop}$ and $M \in \mathfrak{M}_\pi$ the nonsingularity of $\mathfrak{C}\Gamma$ is generic.*

Proof The matrix $\mathcal{C}\Gamma$ is singular if and only if $\det(\mathcal{C}\Gamma) = 0$. Also, $\mathcal{C}\Gamma$ depends linearly on Γ_0. This can be seen from this representaion of $\mathcal{C}\Gamma$ (see Equation (6.2))

$$\mathcal{C}\Gamma = \mathcal{C}\left[\Pi \otimes \Gamma_0\right] \begin{bmatrix} I \\ M \\ \vdots \\ M^{n-1} \end{bmatrix}$$

Thus $\det(\mathcal{C}\Gamma)$ will be a polynomial in the entries of Γ_0. Let

$$g = (\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N, \quad N = mn$$

denote the N-tuple consisting of the entries of Γ_0. Define the polynomial

$$\varphi(\gamma_1, \ldots, \gamma_N) \triangleq \det(\mathcal{C}\Gamma)$$

and the variety

$$V \triangleq \{g \in \mathbb{R}^N \,|\, \varphi(\gamma_1, \ldots, \gamma_N) = 0\}$$

Now $\mathcal{C}\Gamma$ is singular if and only if Γ_0 is such that $g \in V$. The variety V is non-trivial, since any choice such that (Γ_0, M) is not observable results in a singular $\mathcal{C}\Gamma$ according to Lemma 6.1. One such choice is $\Gamma_0 = vq^T$, where $v \in \mathbb{R}^m$ and q^T is a left eigenvector of M. Finally, Theorem 5.3 guarantees the existence of Γ_0 such that $\mathcal{C}\Gamma$ is nonsingular, *i.e.*, $\exists g \notin V$, which completes the proof. ■

The concept of varieties and genericity is of course of pure theoretical interest only, and is included here merely to show that $\mathcal{C}\Gamma$ almost always is nonsingular for total pole assignment. A simple example shows how this concept works:

Example 6.2 In Example 5.3 the continuous-time system

$$\frac{d}{dt}x = Ax + Bu, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is considered, when the desired characteristic polynomial is $p(s) = (s+1)^3$, $\pi \in \mathfrak{P}_{cop}$, and the matrix M is chosen as

$$M = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

With

$$\Gamma_0 = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix}$$

the matrix $\mathcal{C}\Gamma$ becomes

$$\mathcal{C}\Gamma = \begin{bmatrix} 2g_{11} & g_{11} + 2g_{12} & 2g_{13} \\ g_{11} & g_{11} + g_{12} & g_{13} \\ g_{21} & g_{21} + g_{22} & g_{23} \end{bmatrix}$$

Let $g = (g_{11}, g_{12}, g_{13}, g_{21}, g_{22}, g_{23}) \in \mathbb{R}^6$ be the 6-tuple of the entries of Γ_0, and let the polynomial $\varphi(g)$ be the determinant of $\mathcal{C}\Gamma$, *i.e.*,

$$\varphi(g) = \det \mathcal{C}\Gamma = g_{11}(g_{11}g_{23} - g_{13}g_{21})$$

A variety V is defined by $\varphi(g)$ to be the set of all points g in $\mathbb{R}^6$ for which $\varphi(g) = 0$. Thus V is the union of the hyperplane $g_{11} = 0$ and the hypersurface $g_{11}g_{23} = g_{13}g_{21}$, both in $\mathbb{R}^6$. If Γ_0 is chosen randomly, *i.e.*, g is chosen randomly in $\mathbb{R}^6$, $\varphi(g) \neq 0$ with probability one and thus g do not belong to V. This means that $\mathcal{C}\Gamma$ is nonsingular for almost any choice of Γ_0. Still, notice that this is only true if the entries of Γ_0 are chosen independently from each other. For instance, if Γ_0 is chosen as a rank one matrix, *i.e.*, $\Gamma_0 = qv^T$ for some arbitrary vectors $q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$ and $v^T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$, then $\varphi(g) = q_1v_1(q_1v_1q_2v_3 - q_1v_3q_2v_1) \equiv 0$. In this case it is the conditions in Lemma 6.1 that fail, since neither A, nor M is cyclic and therefore (A, Bq) can never be controllable, nor can (v^T, M) be observable. ◦

The fact that $\mathcal{C}\Gamma$ is nonsingular for almost every choice of Γ_0 completes the characterization of the parameterization in the total pole assignment case. Yet, as is illustrated in the example above, some care must be taken in the choice of Γ_0. Any chosen Γ_0 should be checked with the conditions in Lemma 6.1.

6.3 Existence for partial pole assignment

As is seen in Section 5.3 the situation is more intricate for the case $\pi \notin \mathfrak{P}_{cop}$. The necessary conditions of Lemma 6.1 are still valid, but care must be taken, not only in the choice of Γ_0, but also in the choice of the matrix M. It is the common poles of the open and the closed loop systems that restricts the choice of M as is shown in the following lemma:

Lemma 6.3 *Let $q^TA = \lambda q^T$ for some $q \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. If $p(s) = (s - \lambda)r(s)$, where $r(s)$ is a monic polynomial of degree $n - 1$, then, regardless of Γ_0, $\mathcal{C}\Gamma$ is singular if $r(M) = 0$.*

Here λ is a common pole of the open and the closed loop systems, and q^T is a (could be more than one, see below) corresponding left eigenvector of A. Since λ may be complex valued, the coefficients of the monic polynomial $r(s)$ are allowed to be complex valued as well, while $p(s)$ has real coefficients. The lemma is not restricted to the case when the open and closed loop systems have one pole in common — if there are more than one common pole the lemma is applicable for each of these. This means that as soon as $r(M) = 0$ for one of the common poles, the matrix $\mathcal{C}\Gamma$ is singular.
The following lemma is needed as an aid for the proof of Lemma 6.3.

Lemma 6.4 *Let the polynomial* $p(s)$ *be defined by Equation* (3.5)*, and let the polynomials* $\chi_i(s)$*,* $i = 0, \ldots, n-1$*, be defined by Equation* (3.18)*. Furthermore, let*

$$r(s) = \sum_{k=1}^{n} \lambda^{k-1} \chi_{n-k}(s) \tag{6.4}$$

where $\lambda \in \mathbb{C}$*. Then* $p(s) = (s-\lambda)r(s)$ *if and only if* $p(\lambda) = 0$*.*

Proof First we notice that

$$r(s) = \sum_{k=1}^{n} \lambda^{n-k} \chi_{k-1}(s)$$

The polynomials $\chi_0(s), \ldots, \chi_{n-1}(s)$ satisfy the recursive formula (see Equation (3.19))

$$\chi_0(s) = 0, \quad \chi_k(s) = s\chi_{k-1}(s) + p_k, \quad i = 1, \ldots, n-1 \tag{6.5}$$

and when this recursion is continued to $i = n$ we obtain $p(s)$, *i.e.*

$$p(s) = s\chi_{n-1}(s) + p_n \tag{6.6}$$

Multiplying $r(s)$ with $(s - \lambda)$ gives

$$\begin{aligned}
(s-\lambda)r(s) = (s-\lambda)\sum_{k=1}^{n} \lambda^{n-k}\chi_{k-1}(s) &= \sum_{k=1}^{n} \lambda^{n-k} s\chi_{k-1}(s) - \sum_{k=1}^{n} \lambda^{n-k+1}\chi_{k-1}(s) = \\
&= s\chi_{n-1}(s) - \lambda^n + \sum_{k=1}^{n-1} \lambda^{n-k}\left(s\chi_{k-1}(s) - \chi_k(s)\right) = \\
&= s\chi_{n-1}(s) + p_n - \left(\lambda^n + \sum_{k=1}^{n-1} p_k\lambda^{n-k} + p_n\right) = p(s) - p(\lambda)
\end{aligned}$$

where the two last identities follow from (6.5) and (6.6). Hence

$$(s-\lambda)r(s) = p(s) - p(\lambda)$$

which proves the lemma. ∎

Proof of Lemma 6.3 Assume that $q^T A = \lambda q^T$ and that $p(\lambda) = 0$, *i.e.*, $p(s) = (s-\lambda)r(s)$ for some polynomial $r(s)$. We have

$$\mathcal{C}\Gamma = \sum_{k=1}^{n} A^{k-1} B\Gamma_0 \chi_{n-k}(M)$$

Premultiplying $\mathcal{C}\Gamma$ with q^T, the left eigenvector of A, yields

$$q^T\mathcal{C}\Gamma = \sum_{k=1}^{n} q^T A^{k-1} B\Gamma_0 \chi_{n-k}(M) = q^T B\Gamma_0 \sum_{k=1}^{n} \lambda^{k-1}\chi_{n-k}(M) = B\Gamma_0 q^T r(M)$$

where the last identity follows from Lemma 6.4. Hence, if $r(M) = 0$, then $q^T\mathcal{C}\Gamma = 0$, and therefore $\mathcal{C}\Gamma$ must be singular. ■

Thus, for the partial pole assignment problem M must be chosen so that $r(M) \neq 0$. However, $r(M) = 0$ is possible if and only if the minimal polynomial of M is a divisor of $r(M)$. This can only be the case when $r(\lambda) = 0$, *i.e.*, when λ is a multiple eigenvalue of M. More exactly, $r(M) = 0$ is equivalent to that M has at least two linearly independent eigenvectors corresponding to λ, *i.e.*, the rank of $\lambda I - M$ is $n-2$ or less. This means that $r(M) \neq 0$ if there is only one eigenvector of M corresponding to λ, even though $r(\lambda) = 0$. On the other hand, $r(M) \neq 0$ is no guarantee for $\mathcal{C}\Gamma$ to be nonsingular, as will become apparent in Lemma 6.5. To see this we first notice that according to Lemma 4.1 the matrix $\mathcal{C}\Gamma$ satisfies the identity

$$A\mathcal{C}\Gamma - \mathcal{C}\Gamma M = p(A)B\Gamma_0 \tag{6.7}$$

Then assume that $q^T A = \lambda q^T$, and that $p(\lambda) = 0$, *i.e.*, that λ is a common pole of the open and closed loop systems, and that q^T is a left eigenvector of A corresponding to λ. Premultiplying (6.7) with q^T yields

$$q^T(A\mathcal{C}\Gamma - \mathcal{C}\Gamma M) = 0 \quad\Longrightarrow\quad q^T\mathcal{C}\Gamma M = \lambda q^T\mathcal{C}\Gamma M \tag{6.8}$$

since $q^T p(A) = p(\lambda)q^T = 0$. The conclusion from (6.8) is that $q^T\mathcal{C}\Gamma$ is a left eigenvector of M corresponding to λ. Regarding $\mathcal{C}\Gamma$ as a linear mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$ gives at hand that the left eigenvectors of A corresponding to λ are mapped onto left eigenvectors of M that correspond to λ. Hence, if A has k, $1 \leq k \leq n$, linearly independent left eigenvectors that correspond to λ, then $\mathcal{C}\Gamma$ can be nonsingular only if M has k or more linearly independent left eigenvectors that correspond to λ — this follows from the Dimension theorem. This can summed up in the lemma below.

Lemma 6.5 *Let λ be a zero of $p(s)$ and $a(s)$. Then $\mathfrak{C}\Gamma$ is singular for all Γ_0 if*

$$\text{rank}\,(\lambda I - M) > \text{rank}\,(\lambda I - A)$$

Proof The result follows from the discussion above. ■

Together Lemma 6.3 and Lemma 6.5 reduces the class of systems for which the parameterization in Theorem 3.1 and Lemma 3.2 is applicable for the partial pole assignment problem. According to Lemma 6.5 it is required that the rank of $\lambda I - M$ is less than or equal to the rank of $\lambda I - A$ for every common pole λ of the open and closed loop systems. On the other hand, Lemma 6.3 requires that $r(M) \neq 0$, which is equivalent to that the rank of $\lambda I - M$ is equal to $n-1$ for every common pole. So if the rank of $\lambda I - A$ is $n-2$ or less, then $\mathfrak{C}\Gamma$ will be singular, no matter how M is chosen. This observation is stated in the following corollary.

Corollary 6.6 *If $p(\lambda) = 0$ and the rank of $\lambda I - A$ is $n-2$ or less, $\mathfrak{C}\Gamma$ is singular, regardless of M and Γ_0.*

Proof Follows from Lemma 6.3 and Lemma 6.5. ■

Thus, for some systems (3.1) with certain choices of $\pi \notin \mathfrak{P}_{cop}$ the class $\mathfrak{L}_\Gamma$ is empty.

6.4 A discussion on the possibility to modify the parameterization

The conclusions from Sections 5.3 and 6.3 is that the parameterization fails for $\pi \notin \mathfrak{P}_{cop}$ because $p(A)$ is singular. In Theorem 3.1 and the proof thereof it is not obvious why $p(A)$ is present in the parameterization. One might ask whether it is possible to replace $p(A)$ with another matrix that is nonsingular for $\pi \notin \mathfrak{P}_{cop}$. If this is possible the similarity transformation interpretation used in Section 5.2 might be used to prove that $\mathfrak{L}_\Gamma = \mathfrak{L}_\pi$ also for $\pi \notin \mathfrak{P}_{cop}$. However, in what follows it will be shown that this modification of the parameterization in Theorem 3.1 is not possible.

The proof of Theorem 3.1 makes use of Property 3.1, which indicates that a tentative alternative to $p(A)$ should be a polynomial in A. Property 3.1 holds for any monic polynomial of degree n. Thus, let $r(s)$ be an arbitrary monic polynomial of degree n. Let K_ρ be the right companion matrix associated with $r(s)$. If K_ρ and $r(A)$ are used instead of K_π and $p(A)$ in Theorem 3.1, the proof thereof is still applicable. That is, replace Equation (3.13) with

$$\left[K_\rho \otimes I_m\right] \Gamma = \Gamma M \tag{6.9}$$

and Equation (3.15) with

$$L = \left[e_n^T \otimes I_m\right] \Gamma(\mathcal{C}\Gamma)^{-1} r(A) \tag{6.10}$$

Still $\det(sI - M) = p(s)$ must hold.
If $\mathcal{C}\Gamma$ is nonsingular, then $\det(sI - A + BL) = p(s)$, *i.e.*, the desired closed loop poles are obtained.
However, $\mathcal{C}\Gamma$ is nonsingular only for $r(s) = p(s)$, which is stated in the following lemma:

Lemma 6.7 *Let $\mathcal{C}$ be the controllability matrix for the controllable pair (A, B), and let Γ satisfy Equation (6.9). Then $\mathcal{C}\Gamma$ is nonsingular only if*

$$r(s) = p(s) = \det(sI - M)$$

Proof The structure of Γ is given by Lemma 3.2 with some minor modifications; the sequence of polynomials

$$\chi_1(s), \ldots, \chi_{n-1}(s)$$

must be based on the coefficients of $r(s)$. Furthermore, the condition

$$\Gamma_0 r(M) = 0 \tag{6.11}$$

must be satisfied.
Assume that $\Gamma_0 r(M) = 0$ and that $p(\lambda) = 0$ but $r(\lambda) \neq 0$. Consequently λ is an eigenvalue of M, so there exists a vector $q \neq 0$ such that $Mq = \lambda q$. Thus

$$r(M)q = r(\lambda)q \neq 0 \quad \Longrightarrow \quad \Gamma_0 q = \frac{1}{r(\lambda)} \Gamma_0 r(M) q = 0$$

This is recognised as the PBH eigenvector test for observability (see Theorem B.4 in Appendix B). Since $Mq = \lambda q$ and $\Gamma_0 q = 0$ for $q \neq 0$, (Γ_0, M) is not an observable pair. According to Lemma 6.1 observability for (Γ_0, M) is a necessary condition for nonsingularity of $\mathcal{C}\Gamma$. Thus, under the assumptions $\mathcal{C}\Gamma$ is singular.
■

The conclusion is that there is no obvious way to modify the parameterization in order to improve it for $\pi \notin \mathfrak{P}_{cop}$ — $p(A)$ is not that easily replaced.

Chapter 7

A case study

In this chapter a simple model of the Swedish military aircraft JAS Gripen will be used to illustrate the results presented in the previous chapters. The model is based on Examples 2.3, 3.3, 5.4 and 9.1 in Glad and Ljung [18].

7.1 Model description and preliminaries

In normal flight it is a good approximation to disregard from the coupling between pitch angle and turning. The model used here represents the turning dynamics, *i.e.*, the behaviour of the aircraft in the horizontal plane. The course of the aircraft can be influenced by exposing it for a turning torque and/or by a roll. The turning torque is caused by the rudder on the tail fin, while the ailerons on the wings bring the aircraft to roll.
In Figure 7.1 the angles and directions in the model are introduced.
Here

- ψ = the course angle (angle between aircraft body and given course)
- ϕ = the roll angle (around the speed vector)
- β = the side slip angle

The state vector is chosen as $x = \begin{bmatrix} v_y & p & r & \phi & \psi & \delta_a & \delta_r \end{bmatrix}^T$ where

- v_y = the velocity across the aircraft body in the y-direction (for small β $v_y \approx \beta v_0$, where v_0 is the velocity)
- p = roll angular rate in a body-fixed system
- r = turning angular rate in a body-fixed system
- δ_a = the aileron deflection
- δ_r = the rudder deflection

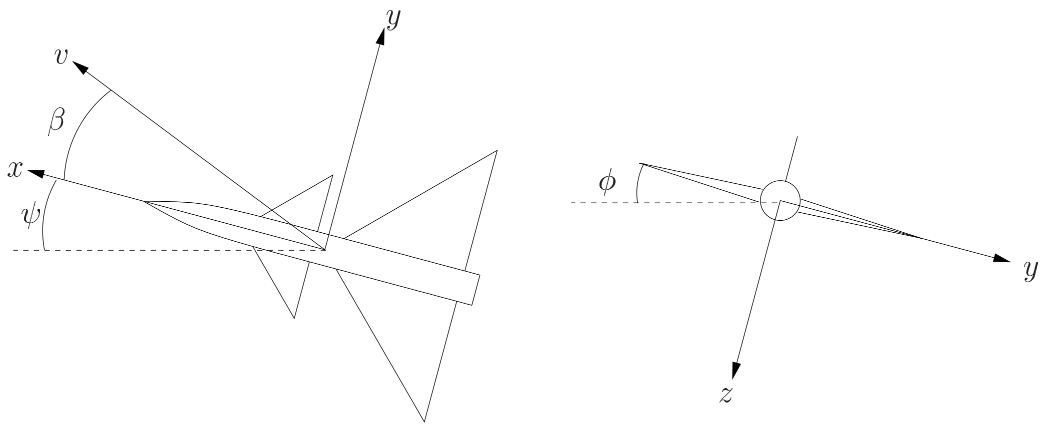


Figure 7.1: Angles for description of the motion of the aircraft. The xyz-axes mark a fixed body coordinate system, and v is the velocity vector of the aircraft. The dashed lines represent a given course (typically north) and the horizontal plane respectively.

It should be noted that p and r are the angular rates around the x- and z-axis, respectively, see Figure 7.1.
The input signals are the commanded aileron and rudder angles

$$u = \begin{bmatrix} \delta_a^{cmd} \\ \delta_r^{cmd} \end{bmatrix} \tag{7.1}$$

and the roll and course angles are regarded as the output signals

$$y = \begin{bmatrix} \phi \\ \psi \end{bmatrix} \tag{7.2}$$

There are also wind disturbances included in the model, represented by the vector

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \tag{7.3}$$

where the decomposition is such that

- w_1 = wind speed across the aircraft (in the y-direction)
- w_2 = wind angular speed along the x-axis
- w_3 = wind angular speed along the z-axis
- w_4 = wind acceleration across the aircraft (in the y-direction)

A linear state space model of the JAS Gripen aircraft can be written

$$\dot{x} = Ax + Bu + Nw \tag{7.4}$$
$$y = Cx \tag{7.5}$$

For a certain case (attack missiles mounted on the wing tips, Mach 0.6, altitude 500 m, angle of attack 0.04 rad) the system matrices are

$$A = \begin{bmatrix} -0.292 & 8.13 & -201 & 9.77 & 0 & -12.5 & 17.1 \\ -0.152 & -2.54 & 0.561 & -0.0004 & 0 & 107 & 7.68 \\ 0.0364 & -0.0678 & -0.481 & 0.0012 & 0 & 4.67 & -7.98 \\ 0 & 1 & 0.0401 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -20 \end{bmatrix} \tag{7.6}$$

$$B = \begin{bmatrix} 0 & -2.15 \\ -31.7 & 0.0274 \\ 0 & 1.48 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix} \tag{7.7}$$

$$N = \begin{bmatrix} 0.292 & 0.001 & 0.97 & 0.0032 \\ 0.152 & -2.54 & 0.552 & 0.000043 \\ -0.0364 & -0.0688 & -0.456 & -0.00012 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.8}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{7.9}$$

All angles are measures in radians and the time unit is seconds.
The open loop system has poles in

$$0, -0.0194, -2.5876, -0.3556 \pm 2.9458i, -20, -20$$

The double pole in -20 corresponds to the servo dynamics of the rudder and the ailerons.

The system is controllable with the controllability indices $\rho_1 = 4$ and $\rho_2 = 3$. There are two linearly independent eigenvectors corresponding to the double pole in -20. This means that A is not cyclic (see Definition A.5 in Appendix A). This rules out the possibility of performing pole assignment using any "single input technique", *i.e.*, by choosing $u(t) = qv(t)$ as input signal for some $q \in \mathbb{R}^2$, where $v(t)$ is a new, scalar input. This follows since, for noncyclic matrices A, the matrix pair (A, Bq) will be uncontrollable for all $q \in \mathbb{R}^2$. That A is not cyclic also restricts the usability of the parameterization in Theorem 3.1 and Lemma 3.2 for the partial pole assignment problem. More precisely; -20 cannot belong to the set of desired poles of the closed loop system, since the rank of $-20I - A$ is five and, according to Corollary 6.6, $\mathfrak{C}\Gamma$ then is singular regardless of Γ_0 and M.

In Example 9.1 in Glad and Ljung [18] the LQ technique is used to obtain a state feedback gain matrix for (7.4). That is, the performance index

$$V = \int_0^\infty \left(x^T Q_1 x + u^T Q_2 u \right) dt \tag{7.10}$$

with

$$Q_1 = C^T C \quad \text{and} \quad Q_2 = 0.1I \tag{7.11}$$

is minimized. This yields the feedback gain

$$L_{LQ} = \begin{bmatrix} -0.0036 & 0.3900 & 0.2098 & 3.1066 & 0.6336 & 1.5386 & 0.0463 \\ -0.0073 & 0.0846 & -0.7750 & 0.5675 & -3.0982 & 0.1968 & 0.2996 \end{bmatrix} \tag{7.12}$$

The LQ solution gives the closed loop system poles

$$-0.1939, -2.6908 \pm 4.0661i, -9.5426 \pm 11.2867i, -20.0282 \text{ and } -21.9014$$

and the corresponding characteristic polynomial

$$\begin{aligned} p(s) = s^7 + 66.590s^6 + 1.8223 \cdot 10^3 s^5 + 2.7175 \cdot 10^4 s^4 + 2.3002 \cdot 10^5 s^3 + \\ + 9.7605 \cdot 10^5 s^2 + 2.4589 \cdot 10^6 s + 4.4182 \cdot 10^5 \end{aligned} \tag{7.13}$$

In order to illustrate the flexibility offered by the design of the matrix Γ_0 the pole configuration given above will be used in the sequel. Since we consider a total pole assignment problem we can, according to Theorem 5.3 find all possible feedback gains. Furthermore, since all desired poles are distinct, the choice of M is less important. All feedback gains can be parameterized with any $M \in \mathfrak{M}_\pi$. In what follows, three different approaches are used in the design of Γ_0. The effect of a randomly chosen Γ_0 is shown in Figures 7.5 and 7.6. In the other

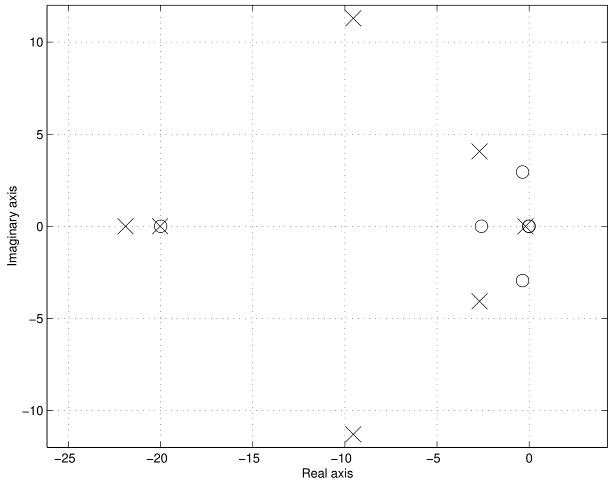


Figure 7.2: The poles of the open loop system are marked with rings and the poles of the closed loop system are marked with $\times$. The open loop system has a double pole in -20, one pole in -0.0194 and one in the origin.

two approaches, which at this stage are preliminary, Γ_0 is chosen such that the knowledge of the initial state values and process disturbances are incorporated. As is well known, the standard LQ solution is indipendent on the initial state value and also from sufficiently well behaved state disturbances, we refrain from a detailed discussion on these matters here.

7.2 A randomized choice of design parameters

To illustrate the significance of the extra degree of freedom offered for multi input systems in the pole assignment problem, a state feedback gain has been generated randomly using the parameterization presented in Chapter 3.3. The matrix M was chosen to $M = A - BL_{LQ}$. A random Γ_0 (obtained with the Matlab function **randn**) was used to generate the following state feedback gain

$$L_1 = \begin{bmatrix} 0.0634 & 0.5020 & -6.0225 & 1.1457 & 3.9550 & 1.0601 & 2.4433 \\ 0.1000 & 0.2418 & -3.2512 & 0.7038 & 2.7728 & 0.3503 & 1.1502 \end{bmatrix} \tag{7.14}$$

Notice that, according to Theorem 6.2, by choosing Γ_0 randomly, $\mathcal{C}\Gamma$ is almost guaranteed to be nonsingular.

As mentioned above, the double pole in -20 prohibits the parameterization of Chapter 3.3 to be used if -20 is also a desired closed loop system pole, since the matrix $\mathcal{C}\Gamma$ always is singular in that case. Here -20.0282 is a desired pole, and while not being exactly -20, it only differs with some 0.1%. This indicates that $\mathcal{C}\Gamma$ may be ill conditioned. Figure 7.3 shows how the condition number of $\mathcal{C}\Gamma$, defined as $\|\mathcal{C}\Gamma\|_2 \cdot \|(\mathcal{C}\Gamma)^{-1}\|_2$, behave when one desired pole is

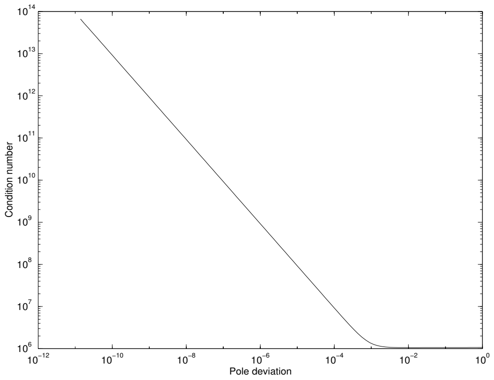


Figure 7.3: The condition number of $\mathcal{C}\Gamma$ with respect to the deviation from -20 for one closed loop system pole.

in the neighbourhood of -20. The diagram is based on the parameters used for L_1. That is, Γ_0 was kept constant, and was chosen as the one used for L_1. The desired poles were also kept constant as those corresponding to the characteristic polynomial (7.13), except for the pole in -20.0282. This pole approached -20 on the real axis. The x-axis in Figure 7.3 represents the distance from the pole to -20. The matrix M was based on the one used to generate L_1, and was only modified to have the correct eigenvalues[1]. The condition number is practically constant for poles further away than 10^{-3} from -20, while it increases rapidly when approaching -20 closer than that. Thus, the desired pole in -20.0282 causes no numerical problems. The relative errors of the poles of the closed loop system compared to the desired poles are of the order 10^{-12} for L_1.

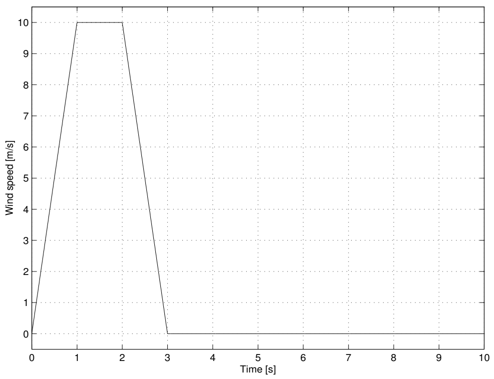
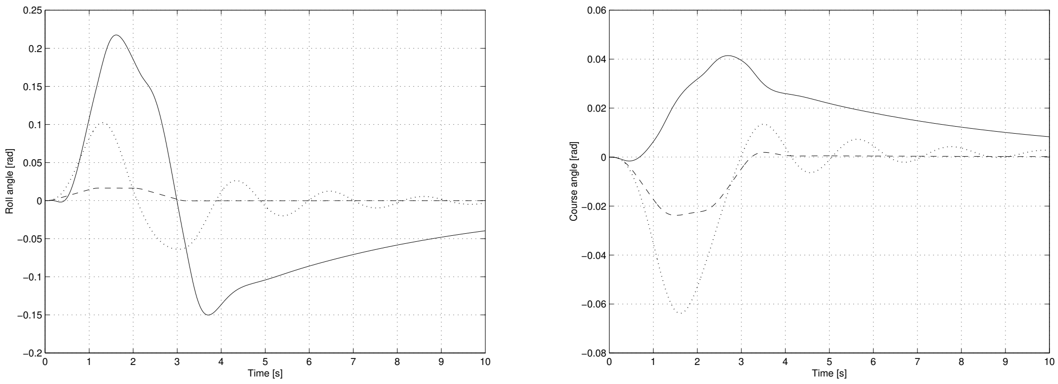


Figure 7.4: The wind speed in the y-direction, w_1, in a test case.

In Example 9.1 in [18] the aircraft model is simulated in a test case, where a wind gust across the aircraft occurs for three seconds. This involves w_1 and w_4,

[1]The matrix was chosen as $M = A - BL_{LQ} + \alpha q v^T / v^T q$, where q and v are the right and left eigenvectors of $A - BL_{LQ}$ respectively corresponding to -20.0282, and α is a scalar.

while the rotational wind speeds, w_2 and w_3 are zero. Figure 7.4 shows how the wind speed w_1 varies with time.
In Figure 7.5 the response on the output signals of the LQ solution L_{LQ} and the randomly generated L_1 is shown for the test case. For comparison purposes the response of the open loop system is also shown. Figure 7.6 shows the corresponding control signals.



(a) (b)

Figure 7.5: The output responses for the test case. Dashed line — L_{LQ}, solid line — L_1, and dotted line — open loop system. Figure (a) shows the roll angle ϕ, and (b) shows the course angle ψ.

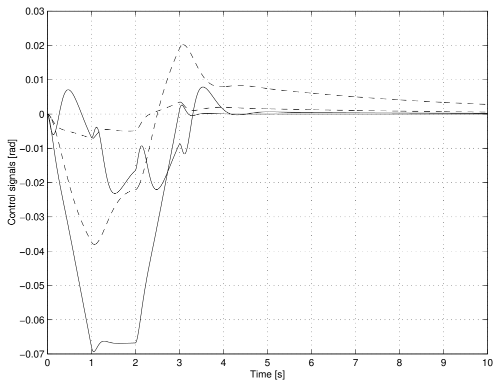


Figure 7.6: The control signals for the test case. Dashed line — L_{LQ}, solid line — L_1.

Regarding the output and control signals, the LQ solution performs much better than the randomly generated state feedback gain. Still the poles of the closed loop systems for L_{LQ} and L_1 are the same. It may even be argued that L_1 performs worse than the open loop system for this case — at least concerning the output signals.

7.3 Design using the initial state values

In this approach we will assume that the initial state is known. The closed loop should be the same as above, *i.e.*, the zeros of the characteristic polynomial (7.13). The performance index (7.10) with

$$Q_1 = C^T \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} C \quad \text{and} \quad Q_2 = 0.1I \tag{7.15}$$

should be minimized. It is clear that the relaxation of the condition on the desired closed loop poles will imply a new standard LQ solution.

$$L'_{LQ} = \begin{bmatrix} -0.0030 & 0.3803 & 0.4583 & 3.0550 & 2.6050 & 1.5370 & -0.0231 \\ -0.0104 & 0.1332 & -1.8562 & 0.7981 & -9.6547 & 0.2228 & 0.6345 \end{bmatrix} \tag{7.16}$$

This would lead to the closed loop poles

$$-0.2097, -5.1882 \pm 6.3891i, -9.5898 \pm 11.3710i, -20.2592 \text{ and } -21.9452$$

and hence this gain is not useful.
The specific initial condition for which the pole assignment gain is optimized will be determined so that the discrepancy in the performance indices is maximized. This is merely made in order to obtain significance in the LQ solutions. The following initial state is obtained.

$$x_0 = \begin{bmatrix} 0.6420 \\ -0.3815 \\ 0.1560 \\ 0.0051 \\ -0.0023 \\ 0.0395 \\ 0.6453 \end{bmatrix} \tag{7.17}$$

In order to solve the optimization problem stated above we will make use of the parameterization considered in the thesis. Since we have a total pole assignment problem we know from Chapter 5 that the optimal gain can be generated from a Γ_0. Due to the explicit parameterization a closed form expression for how the performance index depends on Γ_0 can be derived. It follows that the obtained function is nonlinear and the solution has for this reason been found using the numerical search routine **fminsearch** in Matlab.
The same M was used as for L_1 above, *i.e.*, $M = A - BL_{LQ}$, while the numerical search was performed over (a subset of) $\mathbb{R}^{2\times 7}$ in order to find a suitable Γ_0. The

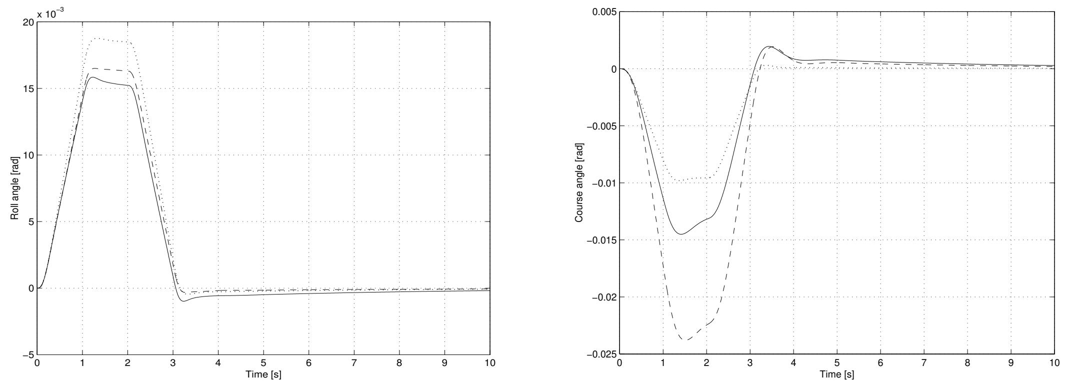
Matlab function **fminsearch** returns a local minimizer near some starting guess. The so obtained solution was

$$\Gamma_0 = \begin{bmatrix} -3.6006 \cdot 10^{-3} & -7.4674 \cdot 10^{-3} \\ 4.0014 \cdot 10^{-1} & 8.5700 \cdot 10^{-2} \\ 2.0877 \cdot 10^{-1} & -7.8067 \cdot 10^{-1} \\ 3.0671 \cdot 10^{0} & 5.7241 \cdot 10^{-1} \\ 6.3399 \cdot 10^{-1} & -3.0975 \cdot 10^{0} \\ 1.5105 \cdot 10^{0} & 1.9720 \cdot 10^{-1} \\ 4.5783 \cdot 10^{-2} & 3.0117 \cdot 10^{-1} \end{bmatrix}^T \tag{7.18}$$

The corresponding feedback gain is

$$L_2 = \begin{bmatrix} -0.0024 & 0.3148 & 0.2246 & 2.2971 & 0.6455 & 1.1421 & -0.0036 \\ -0.0108 & 0.5674 & -0.6257 & 5.8783 & -2.6562 & 2.7901 & 0.5650 \end{bmatrix} \tag{7.19}$$

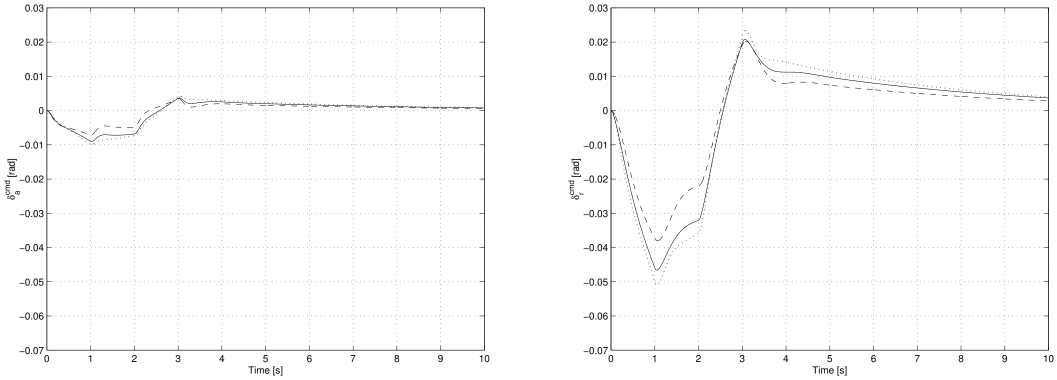
There is no guarantee that the obtained solution is the global minimum of the performance index (most likely it is a local minimum), but at least the value of the performance index (7.10) for L_2 is less than it is for L_{LQ}.



(a) (b)

Figure 7.7: The output responses for the test case. Dashed line — L_{LQ}, solid line — L_2, and dotted line — L'_{LQ}. Figure (a) shows the roll angle ϕ, and (b) shows the course angle ψ.

The aircraft model (7.4) was simulated with $u = -L_2 x$ for the test case desribed in Section 7.2. Figure 7.7 shows the response on the output signals for L_2, and for comparison the responses for L_{LQ} and L'_{LQ} is also shown (remember though that for L'_{LQ} there are other poles). Notice that the scale of the y-axes in Figure 7.7 is different compared to Figure 7.5. However, the curves corresponding to



(a) (b)

Figure 7.8: The control signals for the test case. Dashed line — L_{LQ}, solid line — L_2, and dotted line — L'_{LQ}. Figure (a) shows the commanded aileron angle δ_a^{cmd}, and (b) shows the commanded rudder angle δ_r^{cmd}.

L_{LQ} are exactly the same in Figures 7.5 and 7.7, and they can be used for comparison. The magnitude of the roll angle is smaller for L_2 than for L_{LQ} and L'_{LQ}, while the magnitude of the course angle is between those of L_{LQ} and L'_{LQ}, of which the former is bigger and the latter is smaller. The control signals, shown in Figure 7.7, show a similar pattern, *i.e.*, the magnitude for L_2 lies between those of L'_{LQ} and L_{LQ}. Here the magnitude is smaller for L_{LQ} and bigger for L'_{LQ}.

7.4 Design using knowledge about state disturbances

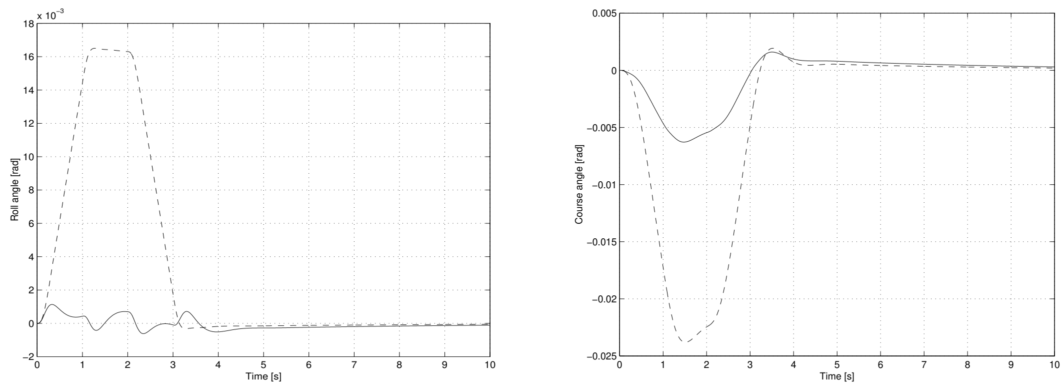
In Section 7.3 the performance index was minimized with respect to the initial state x_0, which was chosen so that the improvement with respect to the performance index potentially should be as big as possible. The test case has no relevance for the optimization problem, but is merely used here for comparison. In fact, there is no guarantee that the true optimal solution to the optimization problem would perform well in the test case. However, assume that the test case is a typical situation, so that during flight this wind disturbance occurs frequently. Then it would be desirable to outperform the disturbance. One plausible approach then is to minimize the performance index with respect to the wind disturbance in the test case. There are of course a number of assumptions that has to be made. Here it is assumed that the initial state is zero, and that the wind gust appears exactly as in Figure 7.4. The matrices Q_1 and Q_2 in (7.10) are as in (7.11), *i.e.*, the same as for the LQ solution L_{LQ}. Since a numerical search has to be employed, the evaluation of the performance index is made by simulation as long as the wind gust is active, and the remainder is determined by the solutio of a Lyapunov equation. Once again the Matlab function **fminsearch** was used, and thus the solution obtained is a local optimum. The same $M = A - BL_{LQ}$ as above was used, and the obtained solution was

$$\Gamma_0 = \begin{bmatrix} -3.8424 \cdot 10^{-3} & -4.0626 \cdot 10^{-3} \\ 5.0550 \cdot 10^{-1} & -6.4236 \cdot 10^{-1} \\ 2.4500 \cdot 10^{-1} & -1.5318 \cdot 10^{0} \\ 3.6920 \cdot 10^{0} & -2.9231 \cdot 10^{1} \\ 1.5704 \cdot 10^{0} & -1.0986 \cdot 10^{1} \\ 1.5484 \cdot 10^{0} & -2.4771 \cdot 10^{0} \\ 7.0509 \cdot 10^{-2} & 5.1728 \cdot 10^{-1} \end{bmatrix}^T \tag{7.20}$$

resulting in the feedback gain matrix

$$L_3 = \begin{bmatrix} -0.0009 & 0.2345 & -1.3962 & 5.2349 & -4.7535 & -0.3826 & 0.4551 \\ -0.0071 & 1.2529 & -10.8134 & 39.4006 & -37.6820 & 4.1982 & 2.7159 \end{bmatrix} \tag{7.21}$$

Figure 7.9 shows the output signals for the test case. Notice that the y-scales



(a) (b)

Figure 7.9: The output responses for the test case. Dashed line — L_{LQ}, solid line — L_3. Figure (a) shows the roll angle ϕ, and (b) shows the course angle ψ.

differ from Figure 7.5, but the curves for L_{LQ} is exactly the same as in Figure 7.5 and thereby it can be used for comparison. The output signals for L_3 are considerably smaller than for L_{LQ}. Particulary for the roll angle the wind disturbance is almost eliminated. To achieve this the control signals of course must be bigger for L_3 than for L_{LQ}, as can be seen in Figure 7.10, but still the control signals for L_3 only exceeds the ones for L_{LQ} with some 50% or so.

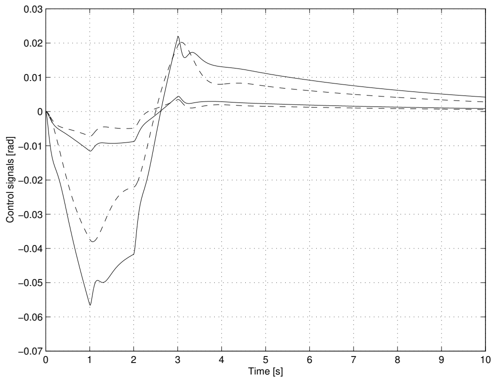


Figure 7.10: The control signals for the test case. Dashed line — L_{LQ}, solid line — L_3.

Chapter 8

Conclusions

In Nordström and Norlander 1997 [41] a parameterization of state feedback gains with respect to pole assignment is presented. In this thesis an extension of this parameterization is derived and characterized in terms of completeness, redundancy and existence.

In the multi-input case there is a degree of freedom in the state feedback gains, and in the analyzed feedback gains this freedom is explicitly defined by the matrix Γ_0, which is of the same dimension as the state feedback gain.

In addition a square matrix M, with eigenvalues that correspond to the desired closed loop polynomial, can be regarded as a design matrix. Hence, the parameterized gains are overparameterized, and it is the subject of the thesis to discuss how this overparameterization can be handled.

The validity of the parameterization, in the sense that a sufficiently large class of gains can be generated is also discussed.

The last item deals with the existence of the gains. As for the single-input case, some variant of the inverse of the controllability matrix must appear in the closed form expression of the feedback gain. Since the controllability matrix is not square in the multi-input case, the inverse of the matrix $\mathcal{C}\Gamma$ is used instead. $\mathcal{C}$ is the controllability matrix, and Γ is a matrix of the same dimension as transposed controllability matrix. The importance of the existence of the inverse of $\mathcal{C}\Gamma$ is discussed in the thesis.

It turns out that the properties of the parameterization depend on whether the open and the closed loop system characteristic polynomials are coprime or not. The case when they are coprime is referred to as total pole assignment, and when they are not it is referred to as partial pole assignment.

In order to make a systematic investigation of the problems mentioned above a classification of different cases is given in Chapter 3.

The Sylvester equation is commonly used when multi input state feedback control problem is considered. In Chapter 4 the proposed parameterization is interpreted in terms of this equation.

The main conclusion of this thesis is that, in the total pole assignment case, the parameterization is complete. That is, all possible feedback gains can be parameterized in this way, as is shown in Chapter 5.

The parameterization includes the inverse of the matrix $\mathcal{C}\Gamma$, where Γ depends on Γ_0 and M. Thus, the question of existence of parameterized feedback gains comes down to the existence of the inverse of $\mathcal{C}\Gamma$. In Chapter 6 some necessary conditions on Γ_0 and M for $\mathcal{C}\Gamma$ to be invertable are given. It is also shown that, for the total pole assignment case, $\mathcal{C}\Gamma$ is invertable for almost every value of Γ_0 and M.

Another conclusion made in the thesis is that the parameterization fails to be complete in the partial pole assignment case. Some of its shortcomings are discussed in Chapter 5. There it is shown how to construct feedback gains that cannot be parameterized in this way. In Chapter 6 some cases where $\mathcal{C}\Gamma$ is singular, regardless of Γ_0 and M, are discussed. These special cases depend on the open loop system and the desired pole location.

In Chapter 5 the influence of the parameters Γ_0 and M is discussed. It is shown that, at least for total pole assignment, M determines the Jordan structure of the closed loop system matrix $(A - BL)$. The "overparameterization", or redundancy, in Γ_0 is also discussed and characterized in this chapter.

Finally, in Chapter 7 a linear model of the military aircraft JAS Gripen has been used to illustrate the results presented in the thesis, and to show how the parametrization can be used in controller design.

Appendix A

Some results from matrix theory

In order to make this thesis self contained a thorough discussion on some relevant theory for matrices is presented in this appendix. Another reason is to give precise definitions of concepts mentioned in the thesis, and which are ambiguously defined in the literature. All results presented here are standard results. Most of the results are given without proof. Proofs can be found in most textbooks in the area, for instance Horn and Johnson 1985 [23].

Definition A.1 Two matrices $F, G \in \mathbb{R}^{n\times n}$ $(\mathbb{C}^{n\times n})$ are *similar* if there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ $(\mathbb{C}^{n\times n})$ such that $TFT^{-1} = G$. ◦

Two similar matrices have exactly identical eigenvalues. This is easily verified by studying the *characteristic polynomial*. Let F and G be similar. According to the definition above there exists a nonsingular matrix T such that $TFT^{-1} = G$. Then $\det(\lambda I - G) = \det(\lambda I - TFT^{-1}) = \det(\lambda I - F)$, *i.e.*, F and G have identical characteristic polynomials, and thus their eigenvalues are identical.

Definition A.2 A *Jordan block* is a matrix $J_i \in \mathbb{C}^{r_i\times r_i}$ with the structure

- $J_i = \lambda$ if $r_i = 1$
- $J_i = \lambda I + I_L^T$ if $r_i > 1$

where $\lambda_i \in \mathbb{C}$, I is the identity matrix and I_L^T is a matrix with ones on the superdiagonal and zeros everywhere else. ◦

Definition A.3 A blockdiagonal matrix $J \in \mathbb{C}^{n\times n}$ of which all diagonal blocks are Jordan blocks is a *Jordan matrix*, or on *Jordan form*. ◦

Theorem A.1 *All matrices $H \in \mathbb{R}^{n\times n}$ $(\mathbb{C}^{n\times n})$ are similar to a Jordan matrix.*

That is, for every matrix $H \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) there exists a matrix $T \in \mathbb{C}^{n\times n}$ such that $J = THT^{-1}$ is a Jordan matrix. The Jordan form is not unique, since any permutation of the Jordan blocks in a block diagonal matrix is a Jordan matrix. However, the set of Jordan blocks is unique for every matrix H, *i.e.*, all Jordan forms of H contains exactly the same Jordan blocks.

Theorem A.2 *Similar matrices have the same Jordan form.*

The scalars λ_i in the Jordan form of a matrix H are identical to the eigenvalues of H. Thus, if H has nonreal eigenvalues, its Jordan form will be complex valued (and so will the transformation matrix T), even though H is real valued. It is sometimes convenient to confine the analysis to real matrices, which motivates the real Jordan form. Such a form is possible, since if a real matrix has a complex valued eigenvalue λ then the complex conjugate $\bar{\lambda}$ is also an eigenvalue of H. Here the stacked bar denotes the complex conjugate, *i.e.*, $\bar{z}$ is the complex conjugate of z. In particular, if, in the Jordan form of H, J_λ is a Jordan block associated to the complex valued eigenvalue λ, then $J_{\bar{\lambda}} = \bar{J}_\lambda$ is also a Jordan block in the Jordan form of H. $J_{\bar{\lambda}}$ is then associated to the eigenvalue $\bar{\lambda}$.
Suppose $H \in \mathbb{R}^{n\times n}$ has the Jordan form J. If all eigenvalues of H are real, J is the real Jordan form of H. If H has the complex valued eigenvalues $\lambda_{1,2} = \mu \pm i\nu$ associated to the Jordan blocks $J_1, J_2(= \bar{J}_1) \in \mathbb{C}^{r\times r}$, these two Jordan blocks are replaced by a real Jordan block. For scalar Jordan blocks, *i.e.*, if $r = 1$, the real Jordan block is $\hat{J}_1 = \Lambda$, where

$$\Lambda = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} \tag{A.1}$$

and if $r > 1$ the real Jordan block is

$$\hat{J}_1 = I_r \otimes \Lambda + I_L^T \otimes I_2 \tag{A.2}$$

where I_r and I_2 are the $r \times r$ and 2×2 identity matrices respectively.
Every eigenvalue of a matrix H has an algebraic and a geometric multiplicity. The *algebraic multiplicity* of an eigenvalue λ is defined as its multiplicity as zero of the characteristic polynomial $p(s)$. That is, if λ has algebraic multiplicity α, then $(s - \lambda)^\alpha$ is a divisor of $p(s)$ but $(s - \lambda)^{\alpha+1}$ is not.
The *geometric multiplicity* of λ is defined as the number of linearly independent eigenvectors of H that correspond to λ. These eigenvectors span a linear subspace. All vectors in this subspace are eigenvectors that correspond to λ. The dimension of this subspace equals the geometric multiplicity.
The algebraic and geometric multiplicities of the eigenvalues of a matrix H can be determined from the Jordan form of H. As mentioned above, every Jordan block is associated to an eigenvalue. There may be more than one Jordan block associated to one eigenvalue. There is exactly one eigenvector corresponding to every Jordan block.

The algebraic multiplicity is equal to the sum of the orders of all Jordan blocks associated to the eigenvalue in question.
The geometric multiplicity is equal to the number of Jordan blocks associated to the eigenvalue.

Theorem A.3 (Caley-Hamilton) *Let $p(\lambda) = \det(\lambda I - H)$ be the characteristic polynomial of the matrix $H \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$). Then $p(H) = 0$.*

This is a very useful result, since it means that, for a matrix $H \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$), any higher power of H can be written as a linear combination of the $n-1$ first powers. One then might ask if there exists any polynomial $a(\lambda)$ of lower degree than n for which $a(H) = 0$.

Definition A.4 The *minimal polynomial* of H is the monic polynomial $a(\cdot)$ of lowest degree that extinguish H, *i.e.*, $a(H) = 0$. ○

The minimal polynomial is unique. A simple example of a matrix whose minimal polynomial has lower degree than its characteristic polynomial is $H = aI$ for any scalar a. The minimal polynomial is $a(\lambda) = \lambda - a$, while the characteristic polynomial is $\det(\lambda I - H) = (\lambda - a)^n$.

Definition A.5 A matrix $H \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$), whose minimal polynomial is identical to its characteristic polynomial, is *cyclic*. ○

In the Jordan form of a cyclic matrix, with Jordan blocks $J_1, \ldots, J_r$ associated to the eigenvalues $\lambda_1, \ldots, \lambda_r$, $\lambda_i \neq \lambda_j$ for $i \neq j$.

Theorem A.4 *Given the matrices $F \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$), $G \in \mathbb{R}^{m\times m}$ ($\mathbb{C}^{m\times m}$) and $H \in \mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) the* Sylvester equation

$$FX - XG = H \tag{A.3}$$

will have a unique solution if and only if F and G have no common eigenvalues.

Proof The proof is based on induction over m.
First, let $m = 1$. Equation (A.3) then can be written as

$$(F - gI)X = H$$

where g is a scalar. If g is an eigenvalue of F the matrix $(F - gI)$ will be singular and the equation will either have infinitely many solutions or none at all. If g is not an eigenvalue of F the matrix $(F - gI)$ will have full rank, and the solution is uniquely given by $X = (F - gI)^{-1}H$.

Now, assume that the theorem is true for $m = p$ and examine the case when $m = p+1$. Take an arbitrary eigenvalue, λ_1, of G, and denote the corresponding eigenvector as q_1. Let $T = \begin{bmatrix} q_1 & g_2 & \cdots & g_{p+1} \end{bmatrix}$ where $g_2, \ldots, g_{p+1}$ are chosen so that T is nonsingular. Use T for a similarity transformation of G that

$$\Omega = T^{-1}GT = \begin{bmatrix} \lambda_1 & z^T \\ 0 & G' \end{bmatrix}$$

where G' has the dimension $p \times p$. Equation (A.3) now can be rewritten as

$$FY - Y\Omega = H' \tag{A.4}$$

where $Y = XT$ and $H' = HT$. The equations (A.3) and (A.4) are equivalent in the sense that (A.4) has a unique solution exactly when (A.3) has a unique solution. Also, Ω has exactly the same eigenvalues as G has. Clearly λ_1 is an eigenvalue of Ω with $e_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T$ as (one) corresponding eigenvector. If Ω has the eigenvalues $\{\lambda_i\}_{i=1}^{p+1}$ ($\lambda_i = \lambda_j$ for $i \neq j$ allowed), then G' will have the eigenvalues $\{\lambda_i\}_{i=2}^{p+1}$, since

$$\det(sI - \Omega) = (s - \lambda_1)\det(sI - G')$$

Denote the columns of Y and H' by y_i and h_i, $i = 1, \ldots, p+1$, respectively. Examining the first column of (A.4) yields $Fy_1 - \lambda_1 y_1 = h_1$. Thus, for the same reasons as above, there is a unique y_1 if and only if λ_1 is not an eigenvalue of F. Examining the other columns yields

$$\begin{aligned} & F\begin{bmatrix} y_2 & \ldots & y_{p+1} \end{bmatrix} - \begin{bmatrix} y_1 & \ldots & y_{p+1} \end{bmatrix} \begin{bmatrix} z^T \\ G' \end{bmatrix} = \begin{bmatrix} h_2 & \ldots & h_{p+1} \end{bmatrix} \\ \iff & F\begin{bmatrix} y_2 & \ldots & y_{p+1} \end{bmatrix} - \begin{bmatrix} y_2 & \ldots & y_{p+1} \end{bmatrix} G' = \begin{bmatrix} h_2 & \ldots & h_{p+1} \end{bmatrix} + y_1 z^T \end{aligned}$$

According to the assumption this equation will have a unique solution if and only if F and G' have no common eigenvalues. Hence, Equation (A.3) will have a unique solution if and only if F and G have no common eigenvalues, and hence, by induction, the theorem is proven. ■

It can be mentioned that if F, G and H in (A.3) are real, there exists a real solution if there exists any solution, for assume that $X = X_{Re} + iX_{Im}$, where $X_{Re}, X_{Im} \in \mathbb{R}^{n\times m}$, is a solution of (A.3). Then $FX_{Re} - X_{Re}G = H$, and thus X_{Re} is a real solution. Furthermore, if F and G have no eigenvalues in common, X_{Re} will be the unique solution of (A.3).
The remainder of this Appendix concerns polynomial matrices. A thorough discussion on polynomial matrices can for instance be found in Kailath 1980 [25].

Definition A.6 A $n \times m$ *polynomial matrix* $P(s)$ is a matrix with entries that are real-coefficient polynomials in s. ∘

In other words, a polynomial matrix is a polynomial in s with coefficients that are real matrices.

Definition A.7 A square polynomial matrix $P(s)$ is *unimodular* if

$$\det P(s) = \text{constant} \neq 0$$

∘

Thus, a unimodular polynomial matrix $P(s)$ has full rank for all $s \in \mathbb{C}$.
Theorem B.7 in Appendix B makes use of a particular form of polynomial matrices called the Smith form.

Definition A.8 The *Smith form* is a $n \times m$ polynomial matrix with the structure

$$\begin{bmatrix} D(s) \\ 0 \end{bmatrix} \quad if \quad n > m, \quad D(s) \text{ if } n = m, \quad \text{and} \quad \begin{bmatrix} D(s) & 0 \end{bmatrix} \text{ if } n < m$$

and where

$$D(s) = \text{diag}\begin{bmatrix} \psi_1(s) & \dots & \psi_r(s) \end{bmatrix}$$

or

$$D(s) = \text{diag}\begin{bmatrix} \psi_1(s) & \dots & \psi_r(s) & 0 & \dots & 0 \end{bmatrix}$$

and where $\psi_1(s), \dots, \psi_r(s)$ are monic polynomials with the property that $\psi_i(\cdot)$ divides $\psi_{i+1}(\cdot)$ for $i = 1, \dots, r-1$. ∘

Theorem A.5 *For any $n \times m$ polynomial matrix $P(s)$ there exist unimodular matrices $U(s)$ and $V(s)$ such that the matrix $\Lambda(s) = U(s)P(s)V(s)$ is on Smith form. $\Lambda(s)$ is then the Smith form of $P(s)$, and it is unique.*

A proof of this result can be found in Kailath 1980 [25].
The polynomials $\psi_1(s), \dots, \psi_r(s)$ in the Smith form of the polynomial matrix $P(s)$ are the *invariant polynomials* of $P(s)$.
A polynomial has a number of zeros in, *i.e.*, there are points $s \in \mathbb{C}$ for which $p(s) = 0$. For a polynomial matrix $P(s)$ the corresponding phenomenon is that $P(s)$ may lose rank for certain $s \in \mathbb{C}$. However, opposite to scalar polynomials, which have at least one zero, polynomial matrices may have the same rank for

all $s \in \mathbb{C}$, like for instance unimodular polynomial matrices. Since unimodular matrices always have full rank, the Smith form $\Lambda(s)$ of a polynomial matrix $P(s)$ will have exactly the same rank as the polynomial matrix $P(s)$ itself for all $s \in \mathbb{C}$. This way the Smith form characterizes the rank properties of a polynomial matrix, just like the Jordan form characterizes the eigenstructure properties of a constant matrix.

Particulary the polynomial matrix $P(s) = sI - H$, where $H \in \mathbb{R}^{n\times n}$, is of interest here since it characterizes the eigenstructure of H. A first degree polynomial matrix, like $P(s)$ here, is also called a *matrix pencil*.

Given the Smith form of $P(s)$, the Jordan form of H can easily be determined, and, equivalently, given the Jordan form of H the Smith form of $P(s)$ is easily obtained. To illustrate this, it will be shown how the Smith form of the matrix pencil $sI - J$, where J is a given Jordan matrix, can be constructed. Let J be the Jordan form of $H \in \mathbb{R}^{n\times n}$. Denote the set of indices $1, \dots, n$ with $i(n)$. Assume that H has $p \leq n$ different eigenvalues, $\lambda_1, \dots, \lambda_p$, where $\lambda_i \neq \lambda_j$ for $i \neq j$. To every λ_i there are q_i Jordan blocks $J^1_{\lambda_i}, \dots, J^{q_i}_{\lambda_i}$ associated, with dimensions $r^j_{\lambda_i} \times r^j_{\lambda_i}$ respectively, and where $r^1_{\lambda_i} \geq \cdots \geq r^{q_i}_{\lambda_i}$. Let

$$\mathfrak{I}(j) = \{i \in i(n) | q_i \geq j\}$$

i.e., the set of i for which there are j or more Jordan blocks associated to λ_i. Let $q^* = \max_i q_i$, then the $n - q^*$ first invariant polynomials of the matrix pencil $sI - J$ will be $\psi_1(s) = \cdots = \psi_{n-q^*}(s) = 1$. The q^* last invariant polynomials are the *nontrivial* invariant polynomials of $sI - J$, and are given by

$$\psi_j(s) = \prod_{i \in \mathfrak{I}(n-j+1)} (s - \lambda_i)^{r^{n-j+1}_{\lambda_i}}, \quad j = n - q^* + 1, \dots, n \tag{A.5}$$

and so the Smith form of $sI - J$ is

$$\Lambda(s) = \operatorname{diag}\begin{bmatrix}1 & \cdots & 1 & \psi_{n-q^*+1}(s) & \cdots & \psi_n(s)\end{bmatrix}$$

and this is of course also the Smith form of $sI - H$. The characteristic polynomial of H is then

$$\det(sI - H) = \det \Lambda(s) = \prod_{j=n-q^*+1}^{n} \psi_j(s)$$

while the minimal polynomial of H is $\psi_n(s)$.

Appendix B

Some results from linear system theory

This thesis exclusively deals with linear time-invariant systems. In this appendix some fundamental theory on linear systems is presented; theory that is extensively used in the thesis. No proofs are given here, but can be found in textbooks on the subject, like for instance Kailath 1980 [25] and Rugh 1996 [46].
Consider a continuous-time linear time-invariant system with state space description

$$\dot{x} = Ax + Bu \tag{B.1}$$

$$y = Cx \tag{B.2}$$

with n states, m inputs and p outputs, *i.e.*,

$$x \in \mathbb{R}^n, \; u \in \mathbb{R}^m, \; y \in \mathbb{R}^p, \; A \in \mathbb{R}^{n\times n}, \; B \in \mathbb{R}^{n\times m} \text{ and } C \in \mathbb{R}^{p\times n}$$

The notions of controllability and observability are fundamental for the understanding of linear systems.

Definition B.1 The state x^* is *controllable* if there exists an input signal that takes the system from the initial state $x(0) = 0$ to x^* in finite time. The system is *controllable* if all states are controllable. ◦

Definition B.2 The state $x^* \neq 0$ is *unobservable* if, when $u(t) \equiv 0$, $t \geq 0$ and $x(0) = x^*$, the output is $y(t) \equiv 0$ for $t \geq 0$. The system is *observable* if it has no unobservable states. ◦

Theorem B.1 *The system* (B.1)–(B.2) *is controllable if and only if the* controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{B.3}$$

has full rank, i.e., rank$\mathcal{C} = n$.

Theorem B.2 *The system* (B.1)–(B.2) *is observable if and only if the* observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{B.4}$$

has full rank, i.e., $\text{rank}\mathcal{O} = n$.

Since the controllability only depends on the matrices A and B, the concept of controllability is sometimes transferred to the matrices themselves; the matrix pair (A, B) is said to be controllable if the corresponding controllability matrix $\mathcal{C}$ has full rank. In the same way, the matrix pair (C, A) is said to be observable if the corresponding observability matrix $\mathcal{O}$ has full rank.
There are other ways of checking the controllability and observability of the matrix pairs (A, B) and (C, A), other than determining the rank of $\mathcal{C}$ and $\mathcal{O}$. Some very useful methods for this are the Popov-Belevitch-Hautus tests, or shorter, the PBH tests.

Theorem B.3 (The PBH eigenvector test for controllability) *The matrix pair (A, B) is controllable if and only if there exists no left eigenvector of A that is orthogonal to all the columns of B,* i.e., *if and only if*

$$q^T A = \lambda q^T \quad \textit{and} \quad q^T B = 0 \quad \Longrightarrow \quad q \equiv 0 \tag{B.5}$$

Theorem B.4 (The PBH eigenvector test for observability) *The matrix pair (C, A) is observable if and only if there exists no right eigenvector of A that is orthogonal to all the rows of C,* i.e., *if and only if*

$$Aq = \lambda q \quad \textit{and} \quad Cq = 0 \quad \Longrightarrow \quad q \equiv 0 \tag{B.6}$$

If the states in the state space description (B.1) are available for measurement, a simple and powerful way to modify the dynamics of the system is to use state feedback. The control law is then

$$u = -Lx, \quad L \in \mathbb{R}^{m \times n} \tag{B.7}$$

where the matrix L is called the state feedback gain. The closed loop system is then

$$\dot{x} = (A - BL)x \tag{B.8}$$

Lemma B.5 (Heymann's lemma [22]) *If the matrix pair (A, B) is controllable, then for every vector $v \in \mathbb{R}^m$ for which $Bv \neq 0$ there exists an $L \in \mathbb{R}^{m \times n}$ such that $(A - BL, Bv)$ is controllable.*

Notice that Bv is a column vector, so the pair $(A - BL, Bv)$ corresponds to a single input system.

Theorem B.6 (Wonham [53]) *When using state feedback, an arbitrary (real-coefficient) characteristic polynomial of the closed loop system* (B.8) *can be obtained if and only if the system* (B.1) *is controllable.*

The theorem states that the closed loop system poles can be chosen arbitrarily — there always exists an L that brings the poles to the desired locations — if the system is controllable. Of course, complex valued poles must occur in conjugated pairs.
Controllability is invariant under state transformation, input substitution and state feedback, *i.e.*, the matrix pair $(T^{-1}(A - BL)T, T^{-1}BG)$ has the same controllability properties as the matrix pair (A, B), for any nonsingular matrices $T \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{m\times m}$ and any $L \in \mathbb{R}^{m\times n}$. This property, and Theorem B.6, can be expressed in a more detailed manner. For this reason the concept of controllability indices is introduced:

Definition B.3 Denote the columns of B as $b_1, \dots, b_m$. For $j = 1, \dots, m$, the jth *controllability index* ρ_j for the matrix pair (A, B) is the smallest integer for which the column vector $A^{\rho_j} b_j$ is linearly dependent on column vectors occurring to the left of it in the controllability matrix $\mathcal{C}$. ◦

The jth controllability index is associated to the jth input. The system (B.1) is controllable if and only if $\sum_{j=1}^{m} \rho_j = n$. The controllability indices are invariant under state transformation, input substitution and state feedback, *i.e.*, the matrix pair $(T^{-1}(A-BL)T, T^{-1}BG)$ has the same set of controllability indices as the matrix pair (A, B), for any nonsingular matrices $T \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{m\times m}$ and any $L \in \mathbb{R}^{m\times n}$. The order of the controllability indices may be changed, for instance by reordering the inputs, but the set of controllability indices will always remain the same.
As mentioned above, Theorem B.6 guarantees the existence of a state feedback gain L such that the desired closed loop poles are obtained if the system is controllable. For controllable systems there is a more detailed result explaining how, apart from the poles, the structure of a system can be modified by state feedback. This result is Rosenbrock's control structure theorem (see Chapter 7.2.2 in Kailath [25]):

Theorem B.7 (Rosenbrock [45]) *Let* (A, B) *be controllable with controllability indices* $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m$. *Let* $\psi_1(s), \dots, \psi_q(s)$, $q \leq m$, *be monic polynomials satisfying*

1. $\psi_{i+1}(s)$ *divides* $\psi_i(s)$ *for* $i = 1, \dots, q-1$

2. $\sum_{i=1}^{q} \deg \psi_i(s) = n$

If and only if

$$\sum_{i=1}^{r} \deg \psi_i(s) \geq \sum_{i=1}^{r} \rho_i \quad \textit{for} \quad r = 1, \ldots, q \tag{B.9}$$

then there exists a matrix $L \in \mathbb{R}^{m \times n}$ such that the matrix pencil $sI - A + BL$ has $\psi_1(s), \ldots, \psi_q(s)$ as its nontrivial invariant polynomials.

See Definition A.8 and Theorem A.5 in Appendix A regarding the Smith form and invariant polynomials.

Proof See for instance Dickinson [15]. ∎

Remark 1 That the controllability indices should satisfy $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m$ is merely included to make the presentation of the result more convenient. This can always be achieved by reordering the inputs, or simply ordering the controllability indices. The result is valid anyway.

Remark 2 Notice that the numbering of the invariant polynomials is reversed compared to the one in Theorem A.5. This is just to simplify the presentation.

Some comments on Rosenbrock's control structure theorem: A consequence of the theorem is that there is an L such that the Smith form of the matrix pencil $sI - A + BL$ is

$$\begin{bmatrix} I_{n-q} & 0 \\ 0 & \Psi(s) \end{bmatrix}, \quad \text{with} \quad \Psi(s) = \operatorname{diag}\begin{bmatrix} \psi_q(s) & \ldots & \psi_1(s) \end{bmatrix} \tag{B.10}$$

In the generic case the controllability indices differ by at most one (Dickinson [15]). According to the theorem the geometric multiplicity can then be chosen to be m for arbitrary many eigenvalues. In degenerated cases $\rho_i \geq \rho_j + 2$ holds for some i and j, and in the extreme case $\rho_1 = n - m + 1$ and $\rho_2 = \cdots = \rho_m = 1$. For these cases the theorem restricts the structure of $A - BL$. Particulary for the latter, extreme case, one eigenvalue can have geometric multiplicity m only if all the other eigenvalues have geometric multiplicity one.

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