The fractional Fourier transform

$$
\begin{aligned}
\texttt{ft}(z)_j &= \sum_{k=0}^{m-1} \exp\Bigl(\frac{-i2\pi jk}{m}\Bigr) z_k, \quad j = 0, \ldots, m-1 \\
\texttt{ift}(z)_j &= \sum_{k=0}^{m-1} \exp\Bigl(\frac{i2\pi jk}{m}\Bigr) z_k, \qquad j = 0, \ldots, m-1
\end{aligned}
$$

If m is a power of two, the `ft` is directly computed by a radix-2 FFT in approximately $5m \log_2 m$ a.o.

We now exploit that $2jk = j^2 + k^2 - (j-k)^2$, which yields

$$
\begin{aligned}
\texttt{ft}(z)_j &= \sum_{k=0}^{m-1} \exp\Bigl(\frac{-i(j^2 + k^2 - (j-k)^2)\pi}{m}\Bigr) z_k \\
&= \exp\Bigl(\frac{-ij^2\pi}{m}\Bigr) \sum_{k=0}^{m-1} \exp\Bigl(\frac{i(j-k)^2\pi}{m}\Bigr) \exp\Bigl(\frac{-ik^2\pi}{m}\Bigr) z_k \\
&= \bar{t}_j \sum_{k=0}^{m-1} t_{j-k} v_k,
\end{aligned}
$$

where

$$
\begin{aligned}
t_j &= \exp\Bigl(\frac{ij^2\pi}{m}\Bigr), \quad j = 0, \ldots, m-1, \\
v_k &= \bar{t}_k z_k, \qquad\qquad k = 0, \ldots, m-1.
\end{aligned}
$$

Unfortunately, the sum is not a circular convolution, i.e., $t_{j-k+m} \neq t_{j-k}$.

Choose $n = 2^{\lceil \log_2 (m-1) \rceil + 1}$, expand t to length n as

$$\begin{aligned} t_j &= 0, & j &= m, \ldots, n-m, \\ t_{n-j} &= t_j, & j &= 1, \ldots, m-1, \\ t_j &= t_{j+n}, & j &< 0, \end{aligned}$$

and pad v with zeros

$$v_k = 0, \qquad k = m, \ldots, n-1.$$

This yields

$$\mathtt{ft}(z)_j = \bar{t}_j \sum_{k=0}^{n-1} t_{j-k} v_k, \quad j = 0, \ldots, m-1.$$

Note that the sum is now a circular convolution of length n, where n is a power of two.

Algorithm FRFT:

$$\begin{aligned} v_j &\leftarrow \bar{t}_j z_j, & j &= 0, \ldots, m-1 \\ v_j &\leftarrow 0, & j &= m, \ldots, n-1 \\ v &\leftarrow \mathtt{ifft}\big(\tfrac{1}{n}\mathtt{fft}(t) \circ \mathtt{fft}(v)\big) \\ \mathtt{ft}(z)_j &\leftarrow \bar{t}_j v_j, & j &= 0, \ldots, m-1 \end{aligned}$$

Algorithm IFRFT:

$$\begin{aligned} v_j &\leftarrow t_j z_j, & j &= 0, \ldots, m-1 \\ v_j &\leftarrow 0, & j &= m, \ldots, n-1 \\ v &\leftarrow \mathtt{ifft}\big(\tfrac{1}{n}\mathtt{fft}(\bar{t}) \circ \mathtt{fft}(v)\big) \\ \mathtt{ift}(z)_j &\leftarrow t_j v_j, & j &= 0, \ldots, m-1 \end{aligned}$$

The algorithms require about $15 n \log_2 n$ a.o. each.

REFERENCES

D. H. Bailey and P. N. Swarztrauber, *The fractional Fourier transform and applications*, SIAM Rev., 33 (1991), pp. 389–404.