Trigonometric Transforms

Assignment: Circulant solver
Global finite-difference formulas

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Problem Formulation

The task is to numerically solve the differential equation

$$u'(x) + u(x) = f(x), \quad 0 < x < 1,$$

subject to the periodic boundary condition

$$u(0) = u(1).$$

To discretize the problem, we introduce a uniform grid of $p+2$ points given by

$$x_j = jh, \quad h = 1/(p+1), \quad j = 0, \dots, p+1.$$

Let u_j denote an approximation of the true solution $u(x_j)$, and order the values in a column vector

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_{p+1} \end{bmatrix}.$$

For the first derivative, a *global*, centered, finite-difference formula is employed:

$$u'(x_j) \approx \sum_{i=1}^{p/2} w_i (u_{j+i} - u_{j-i})/h,$$

where the weights are given by

$$w_i = \frac{(-1)^{i+1}(p/2)!^2}{i(p/2+i)!(p/2-i)!}, \quad i = 1, \dots, p/2.$$

The order of accuracy p is even, and $p \to \infty$ corresponds to the pseudospectral limit. Using the difference formula above and the periodicity condition $u_0 = u_{p+1}$ results in a linear system of equations

$$Cu = b,$$

where

$$b = \begin{bmatrix} hf(x_1) \\ \vdots \\ hf(x_{p+1}) \end{bmatrix}$$

and C is a circulant $(p+1) \times (p+1)$ matrix with first row

$$[h, w_1, \dots, w_{p/2}, -w_{p/2}, \dots, -w_1].$$

Subtask 1

Write a MATLAB function cs(c,b) that solves $Ax = b$, for a circulant matrix A with first row c, by exploiting the fractional Fourier transform. Demonstrate that the function cs is correct by comparing its solution with the solution obtained through Gaussian elimination $A\backslash b$ for some systems of your own choice.

Subtask 2

For the special case $f(x) \equiv \cos(2k\pi x) - 2k\pi \sin(2k\pi x)$, the true solution becomes $u(x) = \cos(2k\pi x)$. Use your circulant solver to determine the lowest value of p *above* spectral resolution $(2k)$, such that $\max_{1 \leq j \leq p+1} |u_j - u(x_j)| < 10^{-5}$ for $k = 20,\ 50,\ 100$. How many grid points per wavelength do the obtained values of p correspond to?