ACOUSTIC SCATTERING FROM SUBMERGED ELASTIC OBJECTS

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Appendix. We now state the partial differential equations and integral equations that govern the strain and stress associated with acoustic scattering from an elastic body immersed in a fluid.

A.1. Fluid media. We consider time-harmonic sound waves in water. Transient acoustic fields would then be handled by Fourier synthesis. Neglecting sound absorption and assuming that the fluid is homogeneous, time-harmonic acoustic waves are governed by the Helmholtz equation

$$\nabla \cdot \nabla p + k^2 p = 0, \tag{A.1}$$

where $p(\boldsymbol{r})$ is the complex pressure, i.e., the scalar phasor [2, p. 337] of the acoustic pressure $\mathrm{Re}(p(\boldsymbol{r})e^{-i\omega t})$ [3, p. 73]. The wavenumber is given by $k = \omega/c_{\mathrm{f}} \equiv 2\pi f/c_{\mathrm{f}}$, where f is the frequency, and c_{f} is the sound speed.

A.2. Solid media. The scatterer is a linear isotropic elastic solid, which we assume is homogeneous. The time-harmonic elastic waves in such a medium are governed by the reduced Navier equations [1, p. 64]

$$\frac{\lambda + 2\mu}{\rho}\nabla\nabla \cdot \boldsymbol{u} - \frac{\mu}{\rho}\nabla \times \nabla \times \boldsymbol{u} + \omega^2 \boldsymbol{u} = 0, \tag{A.2}$$

where $\boldsymbol{u}(\boldsymbol{r})$ is the vector phasor of the displacement $\mathrm{Re}(\boldsymbol{u}(\boldsymbol{r})e^{-i\omega t})$. The density is denoted by ρ, and the Lamé coefficients λ and μ (shear modulus) are related to Poisson's ratio ν and Young's modulus Y by

$$\lambda = \frac{\nu Y}{(1+\nu)(1-2\nu)}, \tag{A.3a}$$

$$\mu = \frac{Y}{2(1+\nu)}. \tag{A.3b}$$

The strain tensor $\boldsymbol{\varepsilon}$ is given by

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\big).$$

It is related to the stress tensor $\boldsymbol{\sigma}$ by the linear constitutive equation

$$\boldsymbol{\sigma} = \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}, \tag{A.4}$$

where $\boldsymbol{I}$ is the identity tensor.

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The Navier equations accommodate two main types of waves. The primary waves (P-waves) are irrotational ($\nabla \times \boldsymbol{u} = 0$), and are also known as dilatation waves. The secondary waves (S-waves) are dilatation-free ($\nabla \cdot \boldsymbol{u} = 0$), and are also known as shear waves. The wave speeds c_p and c_s for P-waves and S-waves, respectively, can be identified from (A.2), which combined with (A.3) yields

$$c_\mathrm{p} \equiv \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}}\sqrt{\frac{Y}{\rho}}, \tag{A.5a}$$

$$c_\mathrm{s} \equiv \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{1}{2(1+\nu)}}\sqrt{\frac{Y}{\rho}}. \tag{A.5b}$$

A.3. Fluid–solid interaction. An accurate model for the acoustic response of the submerged scatterer entails solving (A.1) for the water and (A.2) for the elastic body, and coupling the solutions by appropriate interface conditions [6, p. 29]. For every point on the surface S of the body, the interface conditions on the stress (A.4) and on the displacement are given by

$$\boldsymbol{\tau}(\boldsymbol{u}) \equiv \boldsymbol{\sigma} \cdot \boldsymbol{n} = -p\boldsymbol{n}, \tag{A.6a}$$

$$\boldsymbol{n} \cdot \boldsymbol{u} = (\rho_\mathrm{f}\omega^2)^{-1}\frac{\partial p}{\partial n}, \tag{A.6b}$$

where $\boldsymbol{n}$ is the outward unit normal of S, $\boldsymbol{\tau}$ is the surface traction, and ρ_f is the density of the fluid.

A.4. Boundary integral equations. There are several advantages [4] of using integral representations of the solutions to the three-dimensional Helmholtz equation (A.1) and Navier equations (A.2). For the solution $p(\boldsymbol{r}')$ at any point $\boldsymbol{r}'$ outside the surface S, an integral representation can be derived by employing Green's formula [5, p. 804], resulting in

$$p(\boldsymbol{r}') + \int_S \big(g(\boldsymbol{r},\boldsymbol{r}';k)\frac{\partial p}{\partial n} - \frac{\partial g}{\partial n}(\boldsymbol{r},\boldsymbol{r}';k)p\big)\,dS = p_\mathrm{in}(\boldsymbol{r}'), \tag{A.7}$$

where $g(\boldsymbol{r},\boldsymbol{r}';k)$ is the free-space Green's function [5, p. 805] and p_in is a source term. Since the medium is homogeneous, the Green's function for the Helmholtz equation is known [5, p. 810] explicitly:

$$g(\boldsymbol{r},\boldsymbol{r}';k) = \frac{\exp(ik|\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}.$$

Thus, the solution outside the scatterer could be computed once the solution has been determined on the boundary S. An integral equation for the solution at the boundary is obtained from (A.7) in the limit $\boldsymbol{r}' \to \boldsymbol{r}' \in S$, yielding

$$\tfrac{1}{2}p(\boldsymbol{r}') + \int_S \big(g(\boldsymbol{r},\boldsymbol{r}';k)\frac{\partial p}{\partial n} - \frac{\partial g}{\partial n}(\boldsymbol{r},\boldsymbol{r}';k)p\big)\,dS = p_\mathrm{in}(\boldsymbol{r}'). \tag{A.8}$$

The kernel in (A.7) exhibits an integrable singularity when $\boldsymbol{r}' \to \boldsymbol{r}' \in S$, cf. [6, Sec. 2.5.4]. Hence, special care must be taken in that limit.

The technique can be generalized to the elastic solid case, i.e., the Navier equations (A.2). An integral representation [6, p. 122] for the displacement field is

$$\text{(A.9)} \qquad \boldsymbol{e}_\ell \cdot \boldsymbol{u}(\boldsymbol{r}') + \int_S \big(\boldsymbol{u} \cdot \boldsymbol{\tau}(\boldsymbol{G} \cdot \boldsymbol{e}_\ell) - \boldsymbol{\tau}(\boldsymbol{u}) \cdot (\boldsymbol{G} \cdot \boldsymbol{e}_\ell)\big) dS = 0, \quad \ell = 1, 2, 3,$$

where $\boldsymbol{e}_\ell$ are unit vectors determining the directions of the point source. The symbol $\boldsymbol{G}$ denotes the Green's tensor, whose Cartesian components are

$$G_{ij} = (\rho\omega^2)^{-1}\big(k_{\mathrm{s}}^2 g(\boldsymbol{r}, \boldsymbol{r}'; k_{\mathrm{s}})\delta_{ij} + (g(\boldsymbol{r}, \boldsymbol{r}'; k_{\mathrm{s}}) - g(\boldsymbol{r}, \boldsymbol{r}'; k_{\mathrm{p}}))_{,ij}\big),$$

where k_{p} and k_{s} are the wavenumbers for P-waves and S-waves, respectively. Note that the Green's tensor is built entirely on Green's functions for the scalar Helmholtz equation. By taking the limit $\boldsymbol{r}' \to \boldsymbol{r}' \in S$ for (A.9), the following boundary integral equations are obtained:

$$\text{(A.10)} \quad \tfrac{1}{2}\boldsymbol{e}_\ell \cdot \boldsymbol{u}(\boldsymbol{r}') + \int_S \big(\boldsymbol{u} \cdot \boldsymbol{\tau}(\boldsymbol{G} \cdot \boldsymbol{e}_\ell) - \boldsymbol{\tau}(\boldsymbol{u}) \cdot (\boldsymbol{G} \cdot \boldsymbol{e}_\ell)\big) dS = 0, \quad \ell = 1, 2, 3.$$

Equations (A.8) and (A.10) constitute a pressure–displacement formulation [6, p. 138] of boundary integral equations for scattering from an elastic body immersed in a fluid. The boundary integral equations are coupled by the interface conditions (A.6). Condition (A.6a) is enforced by replacing the surface traction $\boldsymbol{\tau}(\boldsymbol{u})$ in (A.10) with $-p\boldsymbol{n}$. By replacing $\frac{\partial p}{\partial n}$ in (A.8) with $\rho_{\mathrm{f}}\omega^2 \boldsymbol{n} \cdot \boldsymbol{u}$, condition (A.6b) is satisfied.

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