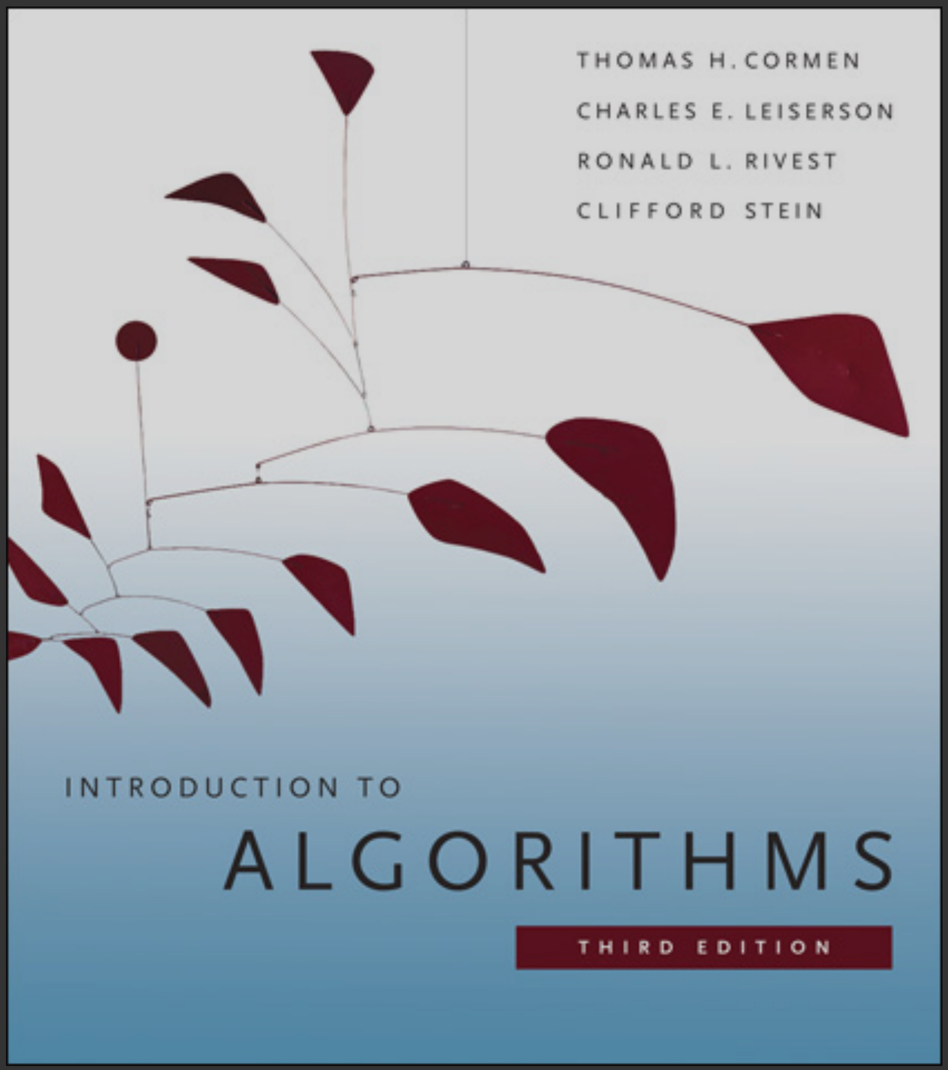

Data structures I, II, III, and IV

I. *Amortized Analysis*
II. *Binary and Binomial Heaps*
III. *Fibonacci Heaps*
IV. *Union–Find*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 2022-01-05 21:35

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Static problems. Given an input, produce an output.

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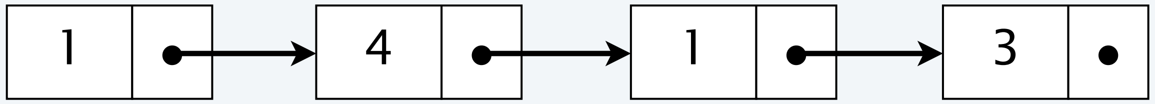

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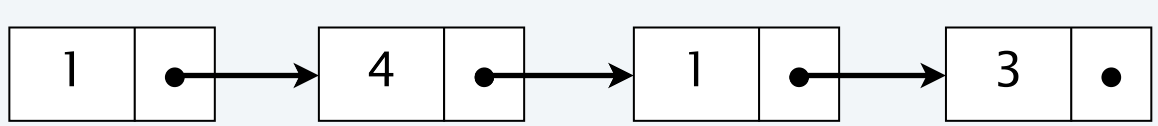

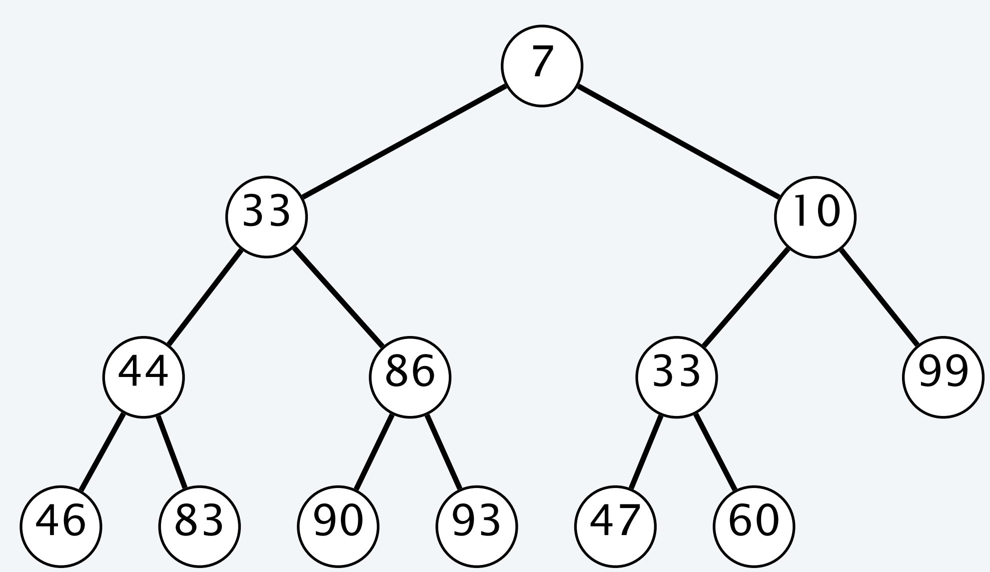

Appetizer

Goal. Design a data structure to support all operations in $O(1)$ time.

- INIT(n): create and return an initialized array (all zero) of length n.
- READ(A, i): return element i in array.
- WRITE(A, i, $value$): set element i in array to $value$.

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true in C or C++, but not Java

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- Given an array, can read or write element i in $O(1)$ time.

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Assumptions.

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- Given an array, can read or write element i in $O(1)$ time.

Remark. An array does $\textsc{Init}$ in $\Theta(n)$ time and $\textsc{Read}$ and $\textsc{Write}$ in $\Theta(1)$ time.

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Data structure. Three arrays $A[1..n]$, $B[1..n]$, and $C[1..n]$, and an integer k.

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A[ ]	?	22	55	99	?	33	?	?

A[4]=99, A[6]=33, A[2]=22, and A[3]=55 initialized in that order

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k = 4

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Theorem. $A[i]$ is initialized iff both $1 \le B[i] \le k$ and $C[B[i]] = i$.

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Pf. Ahead.

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$\text{INIT}\ (A, n)$

$k \leftarrow 0.$

$A \leftarrow \text{MALLOC}(n).$

$B \leftarrow \text{MALLOC}(n).$

$C \leftarrow \text{MALLOC}(n).$

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INIT (A, n)

$k \leftarrow 0.$

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READ (A, i)

IF (IS-INITIALIZED $(A[i])$)

    RETURN $A[i]$.

ELSE

    RETURN 0.

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IF (IS-INITIALIZED $(A[i])$)

  $A[i] \leftarrow value$.

ELSE

  $k \leftarrow k + 1$.

  $A[i] \leftarrow value$.

  $B[i] \leftarrow k$.

  $C[k] \leftarrow i$.

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    $k \leftarrow k + 1.$

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IS-INITIALIZED (A, i)

IF $(1 \le B[i] \le k)$ and $(C[B[i]] = i)$

    RETURN *true*.

ELSE

    RETURN *false*.

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- If $1 \leq B[i] \leq k$ by coincidence, then we still can't have $C[B[i]] = i$ because none of the entries $C[1..k]$ can equal i. ▪

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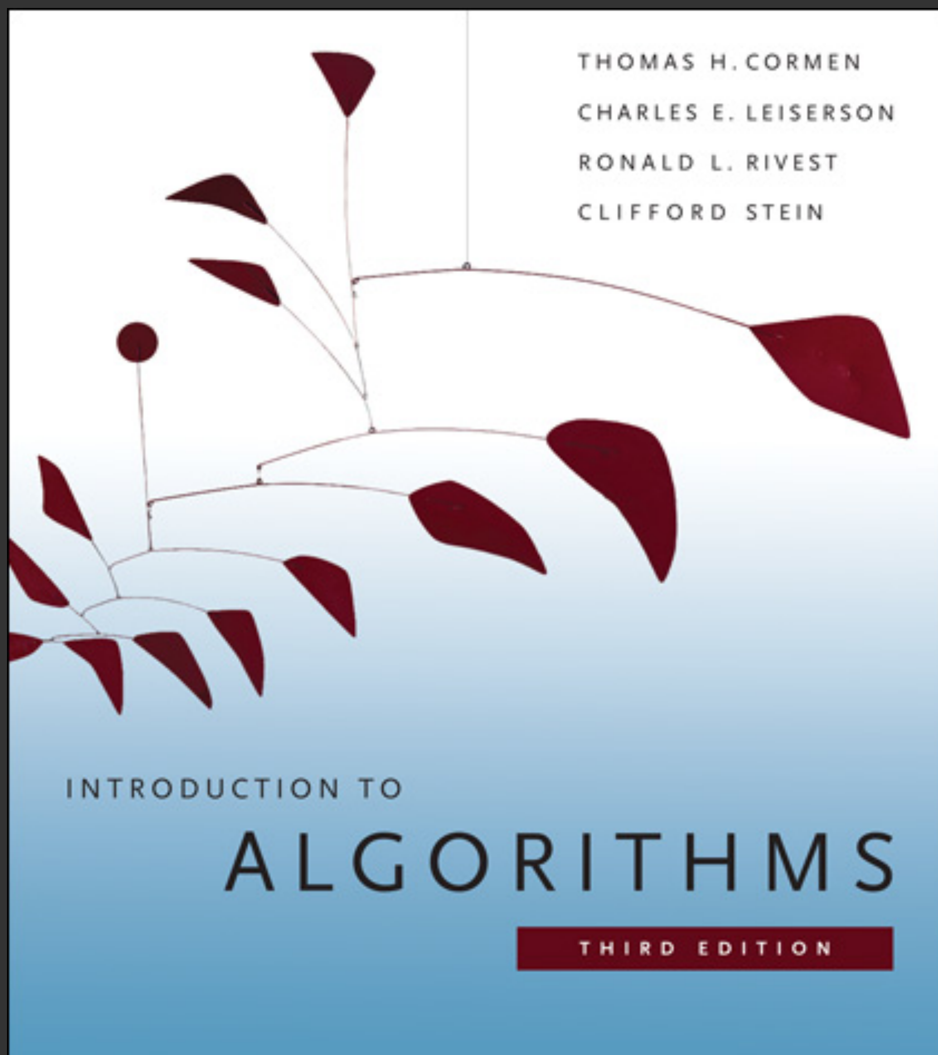

AMORTIZED ANALYSIS

- *binary counter*
- *multi-pop stack*
- *dynamic table*

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Amortized analysis. Determine worst-case running time of a sequence of n data structure operations.

Ex. Starting from an empty stack implemented with a dynamic table, any sequence of n push and pop operations takes $O(n)$ time in the worst case.

Amortized analysis: applications

- Splay trees.
- Dynamic table.
- Fibonacci heaps.
- Garbage collection.
- Move-to-front list updating.
- Push–relabel algorithm for max flow.
- Path compression for disjoint-set union.
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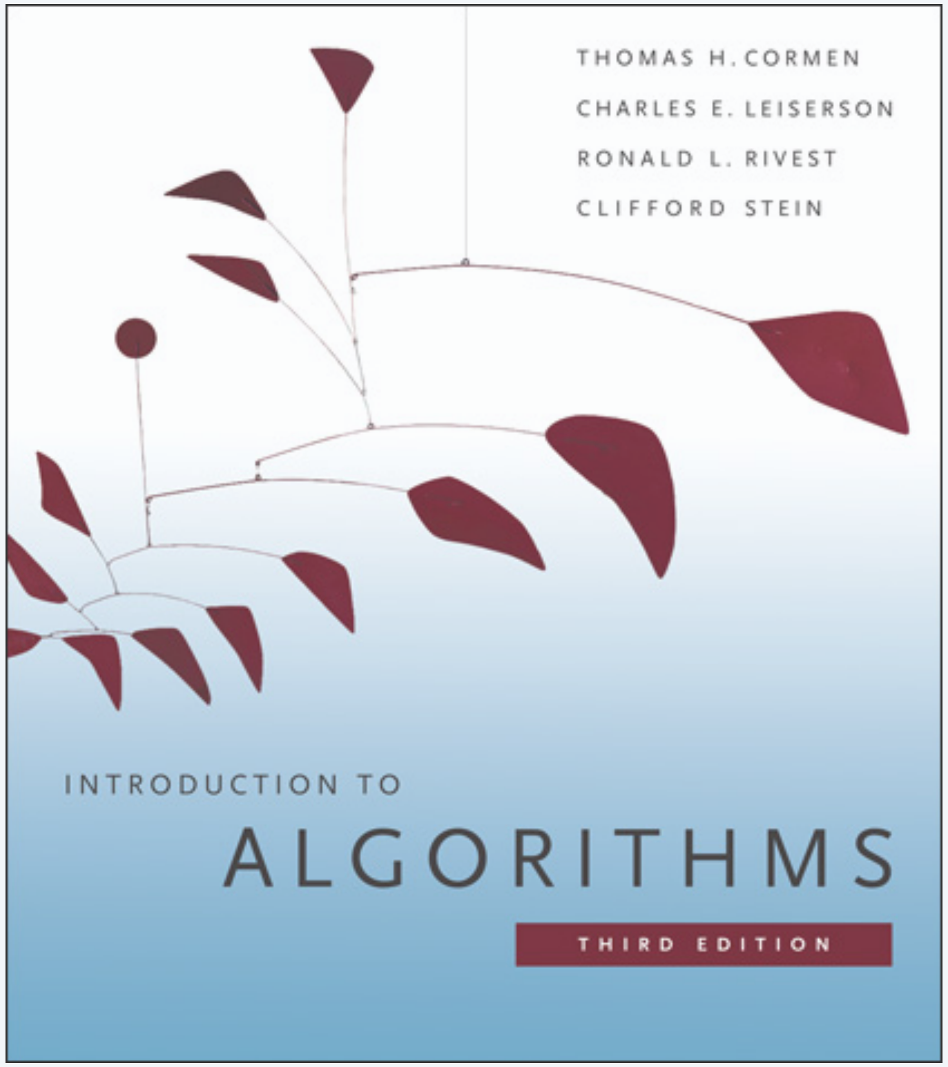
AMORTIZED COMPUTATIONAL COMPLEXITY*

ROBERT ENDRE TARJAN†

Abstract. A powerful technique in the complexity analysis of data structures is *amortization*, or averaging over time. Amortized running time is a realistic but robust complexity measure for which we can obtain surprisingly tight upper and lower bounds on a variety of algorithms. By following the principle of designing algorithms whose amortized complexity is low, we obtain "self-adjusting" data structures that are simple, flexible and efficient. This paper surveys recent work by several researchers on amortized complexity.

ASM(MOS) subject classifications. 68C25, 68E05



CHAPTER 17

AMORTIZED ANALYSIS

- *binary counter*
- *multi-pop stack*
- *dynamic table*

Binary counter

Goal. Increment a k-bit binary counter $(\mathrm{mod}\ 2^k)$.

Representation. $A[j] = j^{th}$ least significant bit of counter.

Counter value	A[7]	A[6]	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	1	1
4	0	0	0	0	0	1	0	0
5	0	0	0	0	0	1	0	1
6	0	0	0	0	0	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	1	0	0	0
9	0	0	0	0	1	0	0	1
10	0	0	0	0	1	0	1	0
11	0	0	0	0	1	0	1	1
12	0	0	0	0	1	1	0	0
13	0	0	0	0	1	1	0	1
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3	0	0	0	0	0	0	1	1
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5	0	0	0	0	0	1	0	1
6	0	0	0	0	0	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	1	0	0	0
9	0	0	0	0	1	0	0	1
10	0	0	0	0	1	0	1	0
11	0	0	0	0	1	0	1	1
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Cost model. Number of bits flipped.

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6	0	0	0	0	0	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	1	0	0	0
9	0	0	0	0	1	0	0	1
10	0	0	0	0	1	0	1	0
11	0	0	0	0	1	0	1	1
12	0	0	0	0	1	1	0	0
13	0	0	0	0	1	1	0	1
14	0	0	0	0	1	1	1	0
15	0	0	0	0	1	1	1	1
16	0	0	0	1	0	0	0	0

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n\,k)$ bits.

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Representation. $A[j] = j^{th}$ least significant bit of counter.

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2	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	1	1
4	0	0	0	0	0	1	0	0
5	0	0	0	0	0	1	0	1
6	0	0	0	0	0	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	1	0	0	0
9	0	0	0	0	1	0	0	1
10	0	0	0	0	1	0	1	0
11	0	0	0	0	1	0	1	1
12	0	0	0	0	1	1	0	0
13	0	0	0	0	1	1	0	1
14	0	0	0	0	1	1	1	0
15	0	0	0	0	1	1	1	1
16	0	0	0	1	0	0	0	0

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n\,k)$ bits.

Pf. At most k bits flipped per increment. ▪

Binary counter

Goal. Increment a k-bit binary counter $(\text{mod } 2^k)$.

Representation. $A[j] = j^{th}$ least significant bit of counter.

Counter value	A[7]	A[6]	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	1	1
4	0	0	0	0	0	1	0	0
5	0	0	0	0	0	1	0	1
6	0	0	0	0	0	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	1	0	0	0
9	0	0	0	0	1	0	0	1
10	0	0	0	0	1	0	1	0
11	0	0	0	0	1	0	1	1
12	0	0	0	0	1	1	0	0
13	0	0	0	0	1	1	0	1
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16	0	0	0	1	0	0	0	0

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n\,k)$ bits. ← overly pessimistic upper bound

Pf. At most k bits flipped per increment. ▪

Aggregate method (brute force)

Aggregate method. Analyze cost of a sequence of operations.

Counter value	A[7]	A[6]	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	Total cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26
16	0	0	0	1	0	0	0	0	31

Binary counter: aggregate method

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Remark. Theorem may be false if initial counter is not zero.

Accounting method (banker's method)

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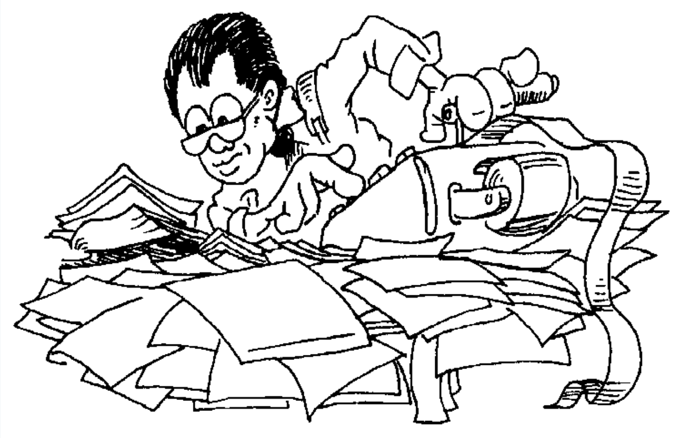

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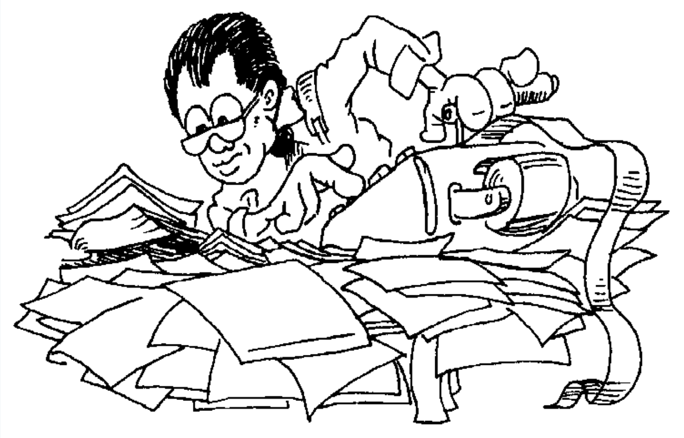

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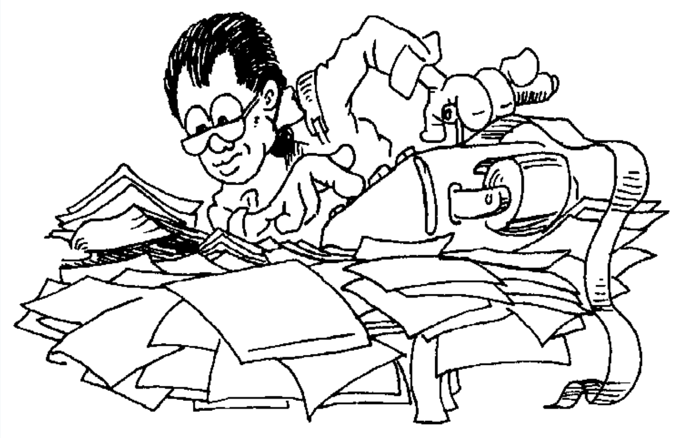

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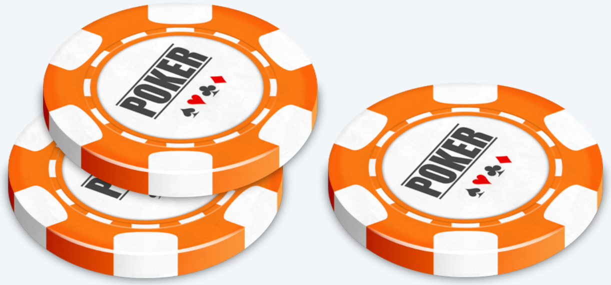

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Credit invariant. The total number of credits in the data structure ≥ 0.

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$$\sum_{i=1} \hat{c}_i - \sum_{i=1} c_i \geq 0$$

our job is to choose suitable amortized costs so that this invariant holds

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credit invariant

Intuition. Measure running time in terms of credits (time = money).

Binary counter: accounting method

Credits. One credit pays for a bit flip.

7	6	5	4	3	2	1	0
0	1	0	0	1	1	1	0

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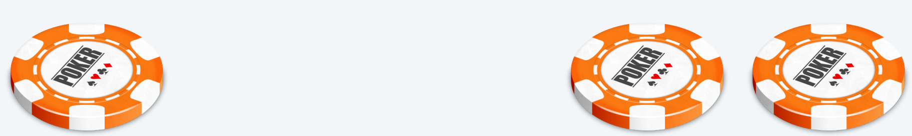

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(unless counter overflows)

- Each INCREMENT operation flips at most one 0 bit to a 1 bit, so the amortized cost per INCREMENT $\leq$ 2.

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- Invariant $\Rightarrow$ number of credits in data structure $\geq$ 0.

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- Flip bit j from 0 to 1: charge 2 credits (use one and save one in bit j).
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Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n)$ bits.

Pf.

the rightmost 0 bit
(unless counter overflows)

- Each INCREMENT operation flips at most one 0 bit to a 1 bit, so the amortized cost per INCREMENT $\leq$ 2.
- Invariant $\Rightarrow$ number of credits in data structure $\geq$ 0.
- Total actual cost of n operations $\leq$ sum of amortized costs $\leq$ $2n$. ▪

accounting method theorem

Potential method (physicist's method)

Potential function. $\Phi(D_i)$ maps each data structure D_i to a real number s.t.:

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each data structure D_i.

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- $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$ = amortized cost of i^{th} operation.

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our job is to choose
a potential function
so that the amortized cost
of each operation is low

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Theorem. Starting from the initial data structure D_0, the total actual cost of any sequence of n operations is at most the sum of the amortized costs.

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Theorem. Starting from the initial data structure D_0, the total actual cost of any sequence of n operations is at most the sum of the amortized costs.

Pf. The amortized cost of the sequence of operations is:

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1}))$$

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$$\begin{aligned}\sum_{i=1}^{n} \hat{c}_i &= \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1})) \\ &= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)\end{aligned}$$

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Binary counter: potential method

7	6	5	4	3	2	1	0
0	1	0	0	1	1	1	0

Binary counter: potential method

Potential function. Let $\Phi(D)$ = number of 1 bits in the binary counter D.

- $\Phi(D_0) = 0$.
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increment

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- Suppose that the i^{th} INCREMENT operation flips t_i bits from 1 to 0.

Binary counter: potential method

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Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n)$ bits.

Pf.

- Suppose that the i^{th} INCREMENT operation flips t_i bits from 1 to 0.
- The actual cost $c_i \leq t_i + 1$. ← operation flips at most one bit from 0 to 1 (no bits flipped to 1 when counter overflows)

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- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$

  $\leq c_i + 1 - t_i$

  potential decreases by 1 for t_i bits flipped from 1 to 0
  and increases by 1 for bit flipped from 0 to 1

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- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$

$$\leq c_i + 1 - t_i$$

← potential decreases by 1 for t_i bits flipped from 1 to 0 and increases by 1 for bit flipped from 0 to 1

$$\leq 2.$$

Binary counter: potential method

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- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$

  $\leq c_i + 1 - t_i$ ← potential decreases by 1 for t_i bits flipped from 1 to 0 and increases by 1 for bit flipped from 0 to 1

  ≤ 2.
- Total actual cost of n operations $\leq$ sum of amortized costs $\leq 2n$. ▪

potential method theorem

Famous potential functions

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Fibonacci heaps. $\Phi(H) \;=\; 2\, trees(H) \;+\; 2\, marks(H)$

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Move-to-front. $\Phi(L) \;=\; 2\, inversions(L, L^*)$

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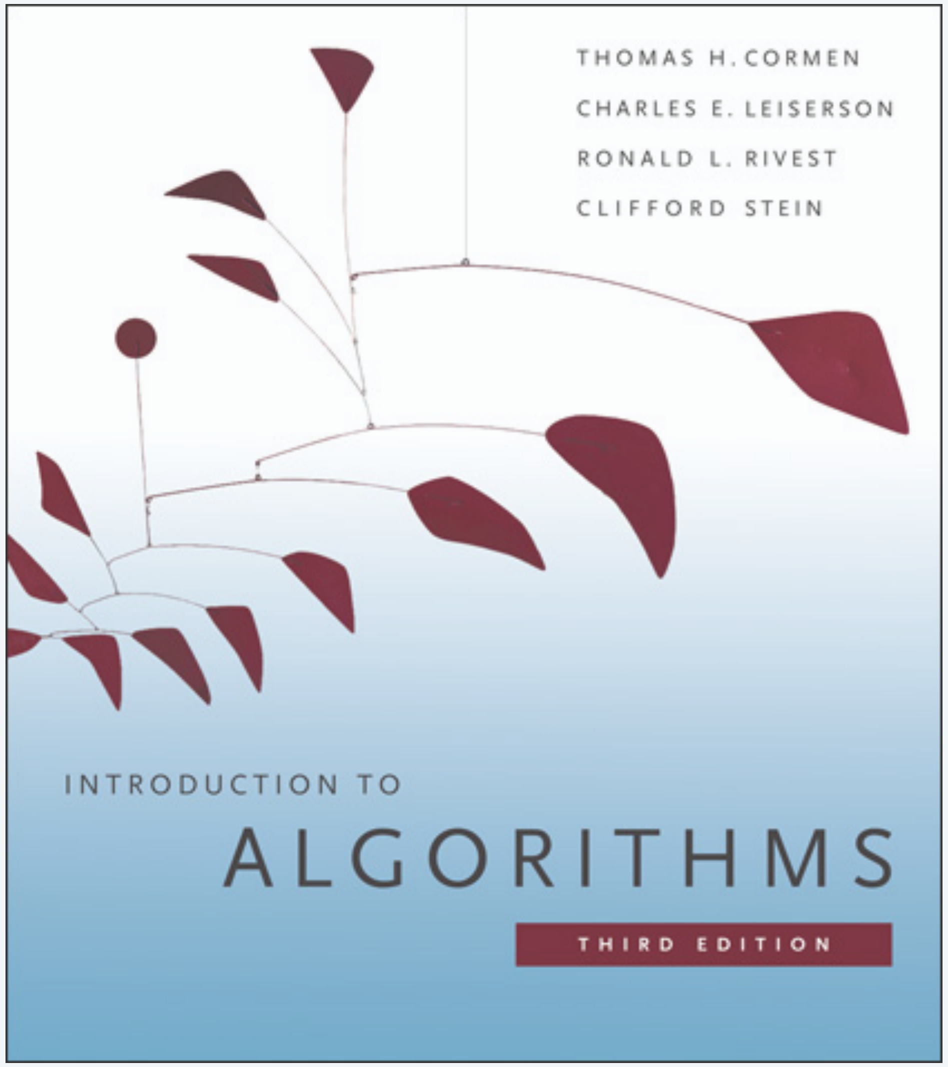
Move-to-front. $\Phi(L) \;=\; 2\, inversions(L, L^*)$

Preflow–push. $\Phi(f) \;=\; \displaystyle\sum_{v \,:\, excess(v) \,>\, 0} height(v)$

Red–black trees. $\Phi(T) \;=\; \displaystyle\sum_{x \in T} w(x)$

$$w(x) \;=\; \begin{cases} 0 & \text{if } x \text{ is red} \\ 1 & \text{if } x \text{ is black and has no red children} \\ 0 & \text{if } x \text{ is black and has one red child} \\ 2 & \text{if } x \text{ is black and has two red children} \end{cases}$$



SECTION 17.4

AMORTIZED ANALYSIS

- *binary counter*
- *multi-pop stack*
- *dynamic table*

Multipop stack

Goal. Support operations on a set of elements:

Multipop stack

Goal. Support operations on a set of elements:

- PUSH(S, x): add element x to stack S.
- POP(S): remove and return the most-recently added element.

Multipop stack

Goal. Support operations on a set of elements:

- PUSH(S, x): add element x to stack S.
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- MULTI-POP(S, k): remove the most-recently added k elements.

```
MULTI-POP(S, k)
─────────────────
FOR i = 1 TO k
    POP(S).
─────────────────
```

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```
MULTI-POP(S, k)
-----------------
FOR i = 1 TO k
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```

Exceptions. We assume POP throws an exception if stack is empty.

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Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n^2)$ time.

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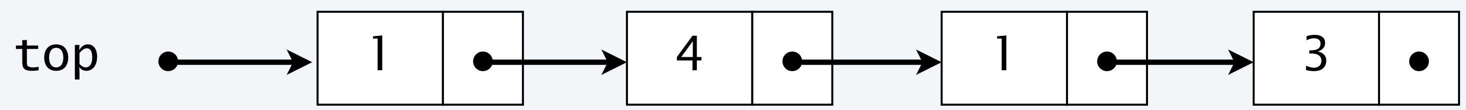
Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n^2)$ time.

Pf.

- Use a singly linked list.



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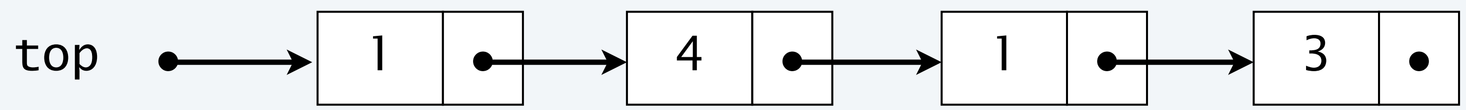
Pf.

- Use a singly linked list.
- POP and PUSH take $O(1)$ time each.



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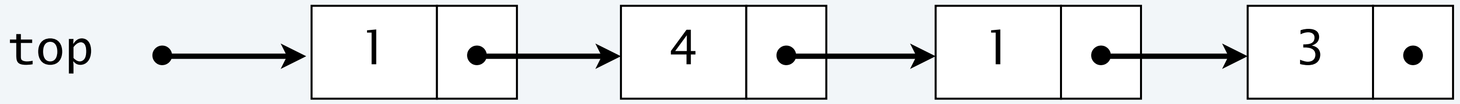
- POP and PUSH take $O(1)$ time each.
- MULTI-POP takes $O(n)$ time. ▪



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Goal. Support operations on a set of elements:

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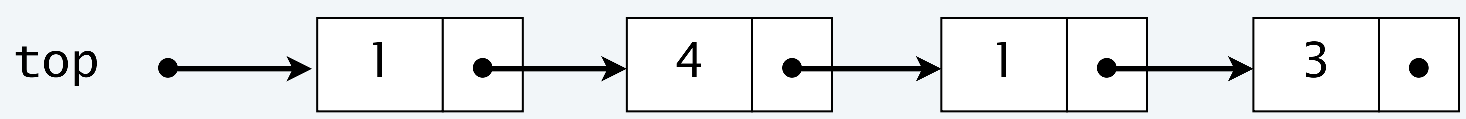
Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n^2)$ time.

overly pessimistic upper bound

Pf.

- Use a singly linked list.
- POP and PUSH take $O(1)$ time each.
- MULTI-POP takes $O(n)$ time. ▪



Multipop stack: aggregate method

Goal. Support operations on a set of elements:

- PUSH(S, x): add element x to stack S.
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Pf.

- An element is popped at most once for each time that it is pushed.

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Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf.

- An element is popped at most once for each time that it is pushed.
- There are $\leq n$ PUSH operations.

Multipop stack: aggregate method

Goal. Support operations on a set of elements:

- PUSH(S, x): add element x to stack S.
- POP(S): remove and return the most-recently added element.
- MULTI-POP(S, k): remove the most-recently added k elements.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf.

- An element is popped at most once for each time that it is pushed.
- There are $\leq n$ PUSH operations.
- Thus, there are $\leq n$ POP operations (including those made within MULTI-POP). ▪

Multipop stack: accounting method

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Credits. 1 credit pays for either a PUSH or POP.

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Invariant. Every element on the stack has 1 credit.

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Accounting.

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- PUSH(S, x): charge 2 credits.

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Accounting.

- PUSH(S, x): charge 2 credits.
  - use 1 credit to pay for pushing x now

Multipop stack: accounting method

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Accounting.

- PUSH(S, x): charge 2 credits.
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  - store 1 credit to pay for popping x at some point in the future

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- PUSH(S, x): charge 2 credits.
  - use 1 credit to pay for pushing x now
  - store 1 credit to pay for popping x at some point in the future
- POP(S): charge 0 credits.

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- PUSH(S, x): charge 2 credits.
  - use 1 credit to pay for pushing x now
  - store 1 credit to pay for popping x at some point in the future
- POP(S): charge 0 credits.
- MULTIPOP(S, k): charge 0 credits.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf.

Multipop stack: accounting method

Credits. 1 credit pays for either a PUSH or POP.

Invariant. Every element on the stack has 1 credit.

Accounting.

- PUSH(S, x): charge 2 credits.
  - use 1 credit to pay for pushing x now
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Pf.

- Invariant $\Rightarrow$ number of credits in data structure ≥ 0.

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Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf.

- Invariant $\Rightarrow$ number of credits in data structure ≥ 0.
- Amortized cost per operation ≤ 2.

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Invariant. Every element on the stack has 1 credit.

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Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf.

- Invariant $\Rightarrow$ number of credits in data structure ≥ 0.
- Amortized cost per operation ≤ 2.
- Total actual cost of n operations $\leq$ sum of amortized costs $\leq 2n$. ▪

accounting method theorem

Multipop stack: potential method

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Potential function. Let $\Phi(D)$ = number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i.

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Pf. [Case 1: push]

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Pf. [Case 1: push]

- Suppose that the i^{th} operation is a PUSH.

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Pf. [Case 1: push]

- Suppose that the i^{th} operation is a PUSH.
- The actual cost $c_i = 1$.

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Pf. [Case 1: push]

- Suppose that the i^{th} operation is a PUSH.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$.

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Pf. [Case 2: pop]

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Pf. [Case 2: pop]

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Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

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- Suppose that the i^{th} operation is a POP.
- The actual cost $c_i = 1$.

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Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

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- Suppose that the i^{th} operation is a POP.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 - 1 = 0$.

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- $\Phi(D_0) = 0$.
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Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [Case 3: multi-pop]

Multipop stack: potential method

Potential function. Let $\Phi(D)$ = number of elements currently on the stack.

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Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [Case 3: multi-pop]

- Suppose that the i^{th} operation is a MULTI-POP of k objects.

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- The actual cost $c_i = k$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = k - k = 0$. ▪

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Pf. [putting everything together]

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Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [putting everything together]

- Amortized cost $\hat{c}_i \leq 2$. ← 2 for push; 0 for pop and multi-pop

Multipop stack: potential method

Potential function. Let $\Phi(D)$ = number of elements currently on the stack.

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- Sum of amortized costs $\hat{c}_i$ of the n operations $\leq 2n$.

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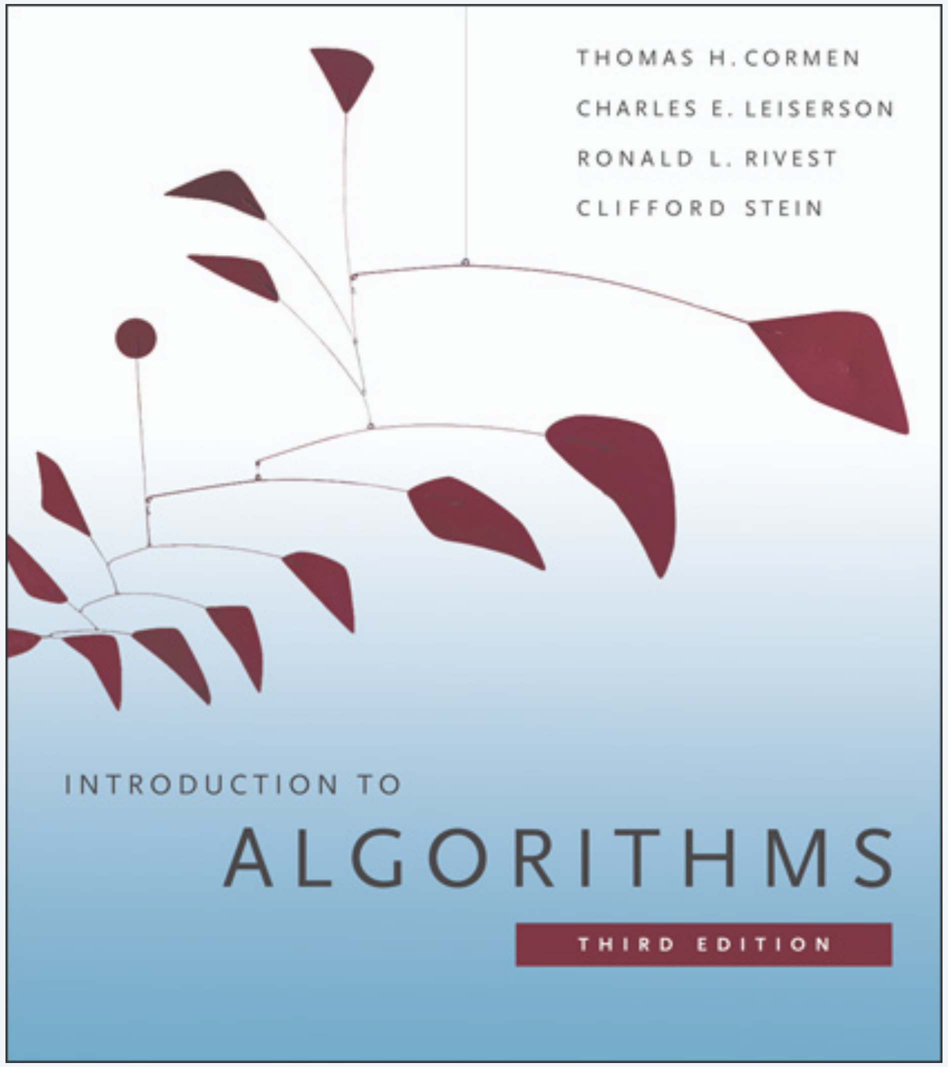
- Sum of amortized costs $\hat{c}_i$ of the n operations $\leq 2n$.
- Total actual cost $\leq$ sum of amortized cost $\leq 2n$. ▪

potential method theorem



SECTION 17.4

AMORTIZED ANALYSIS

- *binary counter*
- *multi-pop stack*
- *dynamic table*

Dynamic table

Goal. Store items in a table (e.g., for hash table, binary heap).

- Two operations: INSERT and DELETE.
  - too many items inserted $\Rightarrow$ expand table.
  - too many items deleted $\Rightarrow$ contract table.
- Requirement: if table contains m items, then space = $\Theta(m)$.

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Theorem. Starting from an empty dynamic table, any intermixed sequence of n INSERT and DELETE operations takes $O(n^2)$ time.

Pf. Each INSERT or DELETE takes $O(n)$ time. ▪

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Theorem. Starting from an empty dynamic table, any intermixed sequence of n INSERT and DELETE operations takes $O(n^2)$ time.

overly pessimistic upper bound

Pf. Each INSERT or DELETE takes $O(n)$ time. ▪

Dynamic table: insert only

- When inserting into an empty table, allocate a table of capacity 1.
- When inserting into a full table, allocate a new table of twice the capacity and copy all items.
- Insert item into table.

insert	old capacity	new capacity	insert cost	copy cost
1	0	1	1	–
2	1	2	1	1
3	2	4	1	2
4	4	4	1	–
5	4	8	1	4
6	8	8	1	–
7	8	8	1	–
8	8	8	1	–
9	8	16	1	8
⋮	⋮	⋮	⋮	⋮

Cost model. Number of items written (due to insertion or copy).

Dynamic table: insert only (aggregate method)

Theorem. [via aggregate method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Dynamic table: insert only (aggregate method)

Theorem. [via aggregate method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let c_i denote the cost of the i^{th} insertion.

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is an exact power of } 2 \\ 1 & \text{otherwise} \end{cases}$$

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Starting from empty table, the cost of a sequence of n INSERT operations is:

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Starting from empty table, the cost of a sequence of n INSERT operations is:

$$\sum_{i=1}^{n} c_i \leq n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j$$

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Starting from empty table, the cost of a sequence of n INSERT operations is:

$$\begin{aligned} \sum_{i=1}^{n} c_i \; &\le \; n \; + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j \\ &< \; n + 2n \end{aligned}$$

Dynamic table: insert only (aggregate method)

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Starting from empty table, the cost of a sequence of n INSERT operations is:

$$\begin{aligned} \sum_{i=1}^{n} c_i \; &\le \; n \; + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j \\ &< \; n + 2n \\ &= \; 3n \quad \blacksquare \end{aligned}$$

Dynamic table demo: insert only (accounting method)

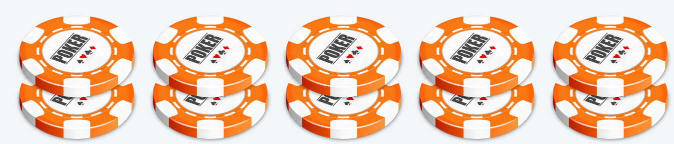
Insert. Charge 3 credits (use 1 credit to insert; save 2 with new item).

Invariant. 2 credits with each item in right half of table; none in left half.

insert N

capacity = 16

A	B	C	D	E	F	G	H	I	J	K	L	M			



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Pf. [by induction]

slight cheat if table capacity = 1
(can charge only 2 credits for first insert)

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Invariant. 2 credits with each item in right half of table; none in left half.

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Pf. [by induction]

- Each newly inserted item gets 2 credits.

Dynamic table: insert only (accounting method)

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- Each newly inserted item gets 2 credits.
- When table doubles from k to $2k$, $k / 2$ items in the table have 2 credits.

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- Each newly inserted item gets 2 credits.
- When table doubles from k to $2k$, $k / 2$ items in the table have 2 credits.
  - these k credits pay for the work needed to copy the k items

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  - now, all k items are in left half of table (and have 0 credits)

Dynamic table: insert only (accounting method)

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Invariant. 2 credits with each item in right half of table; none in left half.

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Theorem. [via accounting method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf.

- Invariant $\Rightarrow$ number of credits in data structure ≥ 0.

Dynamic table: insert only (accounting method)

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Invariant. 2 credits with each item in right half of table; none in left half.

slight cheat if table capacity = 1
(can charge only 2 credits for first insert)

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- Each newly inserted item gets 2 credits.
- When table doubles from k to $2k$, $k / 2$ items in the table have 2 credits.
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  - now, all k items are in left half of table (and have 0 credits)

Theorem. [via accounting method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf.

- Invariant $\Rightarrow$ number of credits in data structure ≥ 0.
- Amortized cost per INSERT = 3.

Dynamic table: insert only (accounting method)

Insert. Charge 3 credits (use 1 credit to insert; save 2 with new item).

Invariant. 2 credits with each item in right half of table; none in left half.

slight cheat if table capacity = 1
(can charge only 2 credits for first insert)

Pf. [by induction]

- Each newly inserted item gets 2 credits.
- When table doubles from k to $2k$, $k / 2$ items in the table have 2 credits.
  - these k credits pay for the work needed to copy the k items
  - now, all k items are in left half of table (and have 0 credits)

Theorem. [via accounting method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf.

- Invariant $\Rightarrow$ number of credits in data structure ≥ 0.
- Amortized cost per INSERT = 3.
- Total actual cost of n operations $\leq$ sum of amortized cost $\leq 3n$. ▪

accounting method theorem

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

1	2	3	4	5	6		

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\ \underset{\text{number of elements}}{\underbrace{size(D_i)}} - \underset{\text{capacity of array}}{\underbrace{capacity(D_i)}}$.

1	2	3	4	5	6		

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\ size(D_i)\ -\ capacity(D_i)$.

number of elements

capacity of array

1	2	3	4	5	6		

size = 6
capacity = 8
Φ = 4

Dynamic table: insert only (potential method)

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Pf. Let $\Phi(D_i) = 2\ size(D_i)\ -\ capacity(D_i)$.

number of elements

capacity of array

- $\Phi(D_0) = 0$.

1	2	3	4	5	6		

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capacity = 8
Φ = 4

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\ \underbrace{size(D_i)}_{\text{number of elements}} - \underbrace{capacity(D_i)}_{\text{capacity of array}}$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i. ← immediately after doubling $capacity(D_i) = 2\ size(D_i)$

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size = 6
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Case 0. [first insertion]

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- Actual cost $c_1 = 1$.

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- $\Phi(D_0) = 0$.
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Case 0. [first insertion]

- Actual cost $c_1 = 1$.
- $\Phi(D_1) - \Phi(D_0) = (2\ size(D_1) - capacity(D_1)) - (2\ size(D_0) - capacity(D_0))$
  $= 1$.

Dynamic table: insert only (potential method)

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- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i.

Case 0. [first insertion]

- Actual cost $c_1 = 1$.
- $\Phi(D_1) - \Phi(D_0) = (2\ size(D_1) - capacity(D_1)) - (2\ size(D_0) - capacity(D_0))$
  $= 1$.
- Amortized cost $\hat{c}_1 = c_1 + (\Phi(D_1) - \Phi(D_0))$
  $= 1 + 1$
  $= 2$.

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\,\underbrace{size(D_i)}_{\text{number of elements}} - \underbrace{capacity(D_i)}_{\text{capacity of array}}$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i.

Case 1. [no array expansion] $capacity(D_i) = capacity(D_{i-1})$.

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\ \underbrace{size(D_i)}_{\text{number of elements}} - \underbrace{capacity(D_i)}_{\text{capacity of array}}$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i.

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- Actual cost $c_i = 1$.

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- $\Phi(D_i) - \Phi(D_{i-1}) = (2\ size(D_i) - capacity(D_i)) - (2\ size(D_{i-1}) - capacity(D_{i-1}))$
  $= 2$.

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  $= 2$.
- Amortized cost $\hat{c}_i = c_i + (\Phi(D_i) - \Phi(D_{i-1}))$
  $= 1 + 2$
  $= 3$.

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- $\Phi(D_i) - \Phi(D_{i-1}) = (2\ size(D_i) - capacity(D_i)) - (2\ size(D_{i-1}) - capacity(D_{i-1}))$
  $= 2 - capacity(D_i) + capacity(D_{i-1})$
  $= 2 - capacity(D_{i-1})$.

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\ size(D_i) - capacity(D_i)$.

number of elements

capacity of array

- $\Phi(D_0) = 0$.
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  $= 2 - capacity(D_{i-1})$.
- Amortized cost $\hat{c}_i = c_i + (\Phi(D_i) - \Phi(D_{i-1}))$
  $= 1 + capacity(D_{i-1}) + (2 - capacity(D_{i-1}))$
  $= 3$.

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Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

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[putting everything together]

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[putting everything together]

- Amortized cost per operation $\hat{c}_i \le 3$.
- Total actual cost of n operations $\underbrace{\le}_{\text{potential method theorem}}$ sum of amortized cost $\le 3n$. ▪

Dynamic table: doubling and halving

Thrashing.

- INSERT: when inserting into a full table, double capacity.
- DELETE: when deleting from a table that is ½-full, halve capacity.

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Efficient solution.

- When inserting into an empty table, initialize table size to 1; when deleting from a table of size 1, free the table.
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Memory usage. A dynamic table uses $\Theta(n)$ memory to store n items.

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Memory usage. A dynamic table uses $\Theta(n)$ memory to store n items.

Pf. Table is always between 25% and 100% full. ▪

Dynamic table demo: insert and delete (accounting method)

Insert. Charge 3 credits (1 to insert; save 2 with item if in right half).

Delete. Charge 2 credits (1 to delete; save 1 in empty slot if in left half).

Invariant 1. 2 credits with each item in right half of table.

Invariant 2. 1 credit with each empty slot in left half of table.

delete M

capacity = 16

A	B	C	D	E	F	G	H	I	J	K	L	M			

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Invariant 1. 2 credits with each item in right half of table. ← to pay for expansion

Invariant 2. 1 credit with each empty slot in left half of table.

Theorem. [via accounting method] Starting from an empty dynamic table, any intermixed sequence of n INSERT and DELETE operations takes $O(n)$ time.

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Pf.

- Invariants $\Rightarrow$ number of credits in data structure ≥ 0.

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accounting method theorem

Dynamic table: insert and delete (potential method)

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Pf sketch.

- Let $\alpha(D_i) = size(D_i) \,/\, capacity(D_i)$.

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- Let $\alpha(D_i) = size(D_i) \;/\; capacity(D_i)$.
- Define $\Phi(D_i) = \begin{cases} 2\, size(D_i) \; - \; capacity(D_i) & \text{if } \alpha(D_i) \geq 1/2 \\ \frac{1}{2}\, capacity(D_i) \; - \; size(D_i) & \text{if } \alpha(D_i) < 1/2 \end{cases}$

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- $\Phi(D_0) = 0,\ \Phi(D_i) \geq 0.$ [a potential function]
- When $\alpha(D_i) = 1/2,\ \Phi(D_i) = 0.$ [zero potential after resizing]

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- When $\alpha(D_i) = 1$, $\Phi(D_i) = size(D_i)$. [can pay for expansion]

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- When $\alpha(D_i) = 1$, $\Phi(D_i) = size(D_i)$. [can pay for expansion]
- When $\alpha(D_i) = 1/4$, $\Phi(D_i) = size(D_i)$. [can pay for contraction]

  …