Propositional Satisfiability (SAT): DPLL

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DLL62: Preliminary definitions

- Propositional variables can be assigned value False or True
  - In some contexts variables may be unassigned
- A clause is satisfied if at least one of its literals is assigned value True

  $(x_1 \vee \neg x_2 \vee \neg x_3)$
- A clause is unsatisfied if all of its literals are assigned value False (also called a conflict clause)

  $(x_1 \vee \neg x_2 \vee \neg x_3)$
- A clause is unit if it contains one single unassigned literal and all other literals are assigned value False

  $(x_1 \vee \neg x_2 \vee \neg x_3)$
- A formula is satisfied if all of its clauses are satisfied
- A formula is unsatisfied if at least one of its clauses is unsatisfied

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DLL62: space efficient DP60

Davis, Martin; Logemann, George, and Loveland, Donald (1962). "A Machine Program for Theorem Proving". Communications of the ACM 5 (7): 394-397.

- Standard backtrack search
- DPLL(F):
  - Apply unit propagation
  - If conflict identified, return UNSAT
  - Apply the pure literal rule
  - If F is satisfied (and possibly empty), return SAT
  - Select unassigned variable x
    - If DPLL($F \wedge x$)=SAT return SAT
    - return DPLL($F \wedge \neg x$)

Proof system: tree resolution

Pure Literals in backtrack search

- Pure literal rule:
  Clauses containing pure literals can be removed from the formula (i.e. just satisfy those pure literals)
- Example:
  $\varphi = (\neg x_1 \vee x_2) \wedge (x_3 \vee \neg x_2) \wedge (x_4 \vee \neg x_5) \wedge (x_5 \vee \neg x_4)$
- The resulting formula becomes:
  $\varphi_{\neg x_1, x_3} = (x_4 \vee \neg x_5) \wedge (x_5 \vee \neg x_4)$
- if ℓ is a pure literal in φ, then $\varphi_\ell \subset \varphi$
- Preserve satisfiability, not logical equivalency!

Unit Propagation in backtrack search

- Unit clause rule in backtrack search:
  Given a unit clause, its only unassigned literal must be assigned value True for the clause to be satisfied
- Example: for unit clause $(x_1 \vee \neg x_2 \vee \neg x_3)$, x_3 must be assigned value False
- Unit propagation
  Iterated application of the unit clause rule

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_4)$$

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Unit propagation can satisfy clauses but can also falsify clauses (i.e. conflicts)

An Example of DPLL

$$
\begin{aligned}
\varphi \quad = \quad & (a \vee \neg b \vee d) \wedge (a \vee \neg b \vee e) \wedge \\
& (\neg b \vee \neg d \vee \neg e) \wedge (\neg a \vee \neg b) \wedge \\
& (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \neg d) \wedge \\
& (a \vee b \vee \neg c \vee e) \wedge (a \vee b \vee \neg c \vee \neg e)
\end{aligned}
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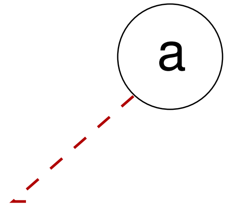
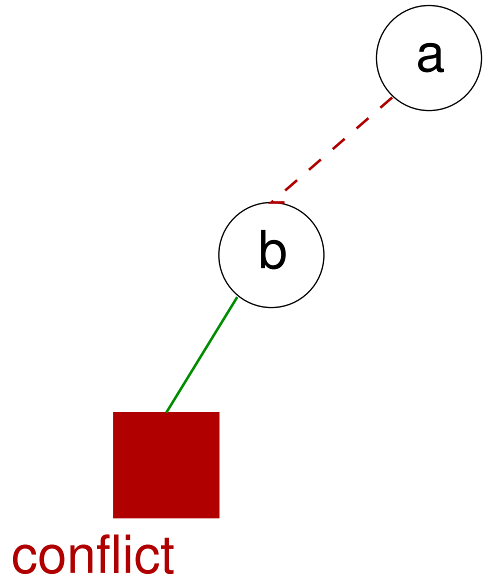


An Example of DPLL

$$\begin{aligned}\varphi \;=\; & (a \lor \neg b \lor d) \land (a \lor \neg b \lor e) \land \\ & (\neg b \lor \neg d \lor \neg e) \land (\neg a \lor \neg b) \land \\ & (a \lor b \lor c \lor d) \land (a \lor b \lor c \lor \neg d) \land \\ & (a \lor b \lor \neg c \lor e) \land (a \lor b \lor \neg c \lor \neg e)\end{aligned}$$



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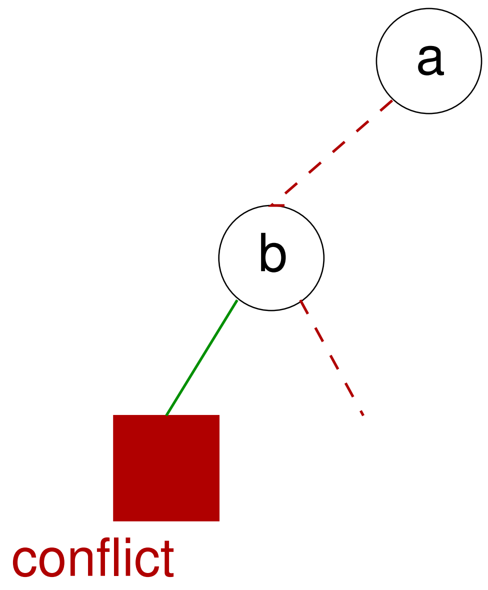

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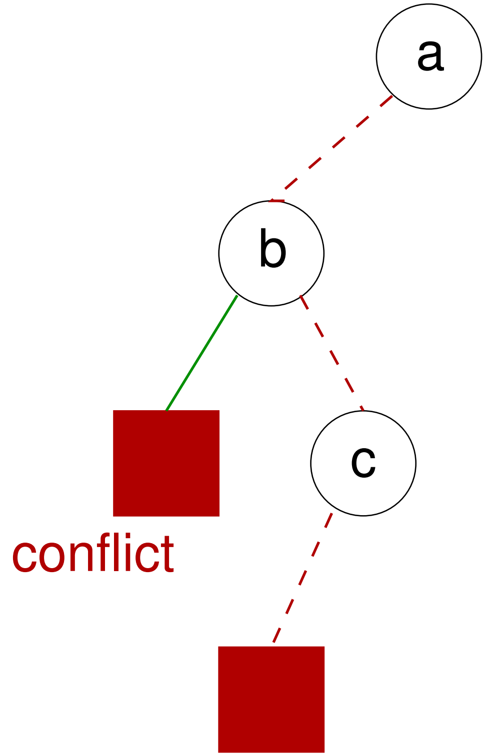

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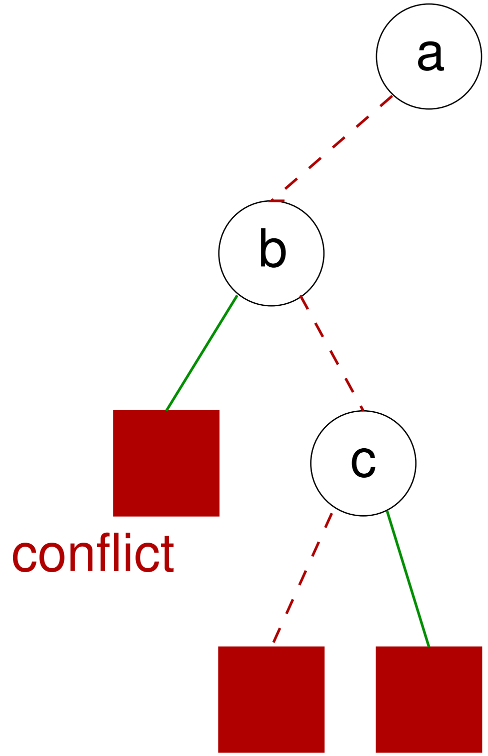

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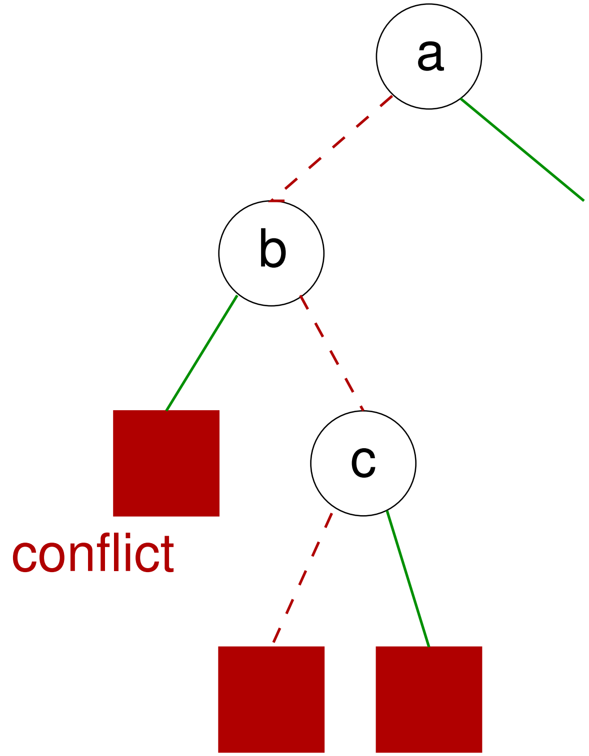

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