Propositional Satisfiability (SAT): Modelling

Tjark Weber

Propositional Formulas

A propositional formula is defined over a set of propositional variables $x_1, x_2, \ldots$, using the standard propositional connectives $\neg$, $\wedge$ and $\vee$.

Example: $(\neg x_1 \vee x_3) \wedge (x_2 \vee x_3) \wedge (\neg x_2 \vee x_3)$

The domain of propositional variables is $\{\mathrm{True}, \mathrm{False}\}$.

A literal is a propositional variable or its negation.

Examples: x_1, $\neg x_2$

A clause is a disjunction of literals.

Example: $\neg x_1 \vee x_3$

Conjunctive Normal Form (CNF)

A formula in conjunctive normal form (CNF) is a conjunction of clauses.

Example: $(\neg x_1 \vee x_3) \wedge (x_2 \vee x_3) \wedge (\neg x_2 \vee x_3)$

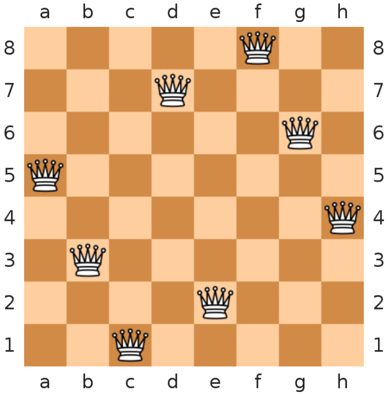
- An empty conjunction of clauses $\bigwedge \emptyset$ is trivially true (satisfied by every assignment).
- An empty clause (disjunction of literals) $\bigvee \emptyset$ is trivially false (satisfied by no assignment).

Notation:

- $\top$ is the trivially true formula.
- $\bot$ is the trivially false formula.
- $\varphi \models \psi$: φ implies ψ

The n-Queens Problem

Given a positive integer n, place n chess queens on an $n \times n$ chessboard so that no two queens are in the same row, column, or diagonal.



Encoding the n-Queens Problem in SAT

Idea:

Construct a propositional formula (in conjunctive normal form, CNF) such that each

satisfying assignment

corresponds to a

solution to the n-queens problem.

Encoding the 2-Queens Problem in SAT

Propositional variables:

Encoding the 2-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 2$).

Encoding the 2-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 2$).

Constraints:

- There is (at least) a queen in row 1:

Encoding the 2-Queens Problem in SAT

Propositional variables:

Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 2$).

Constraints:

- There is (at least) a queen in row 1: $x_{11} \vee x_{12}$
- There is (at least) a queen in row 2:

Encoding the 2-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 2$).

Constraints:

- There is (at least) a queen in row 1: $x_{11} \vee x_{12}$
- There is (at least) a queen in row 2: $x_{21} \vee x_{22}$
- There are no two queens in the same row:

Encoding the 2-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 2$).

Constraints:

- There is (at least) a queen in row 1: $x_{11} \vee x_{12}$
- There is (at least) a queen in row 2: $x_{21} \vee x_{22}$
- There are no two queens in the same row:
  $\neg(x_{11} \wedge x_{12}) \wedge \neg(x_{21} \wedge x_{22})$
  that is in CNF:

Encoding the 2-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 2$).

Constraints:

- There is (at least) a queen in row 1: $x_{11} \vee x_{12}$
- There is (at least) a queen in row 2: $x_{21} \vee x_{22}$
- There are no two queens in the same row:
  $\neg(x_{11} \wedge x_{12}) \wedge \neg(x_{21} \wedge x_{22})$
  that is in CNF: $(\neg x_{11} \vee \neg x_{12}) \wedge (\neg x_{21} \vee \neg x_{22})$
- There are no two queens in the same column:

Encoding the 2-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 2$).

Constraints:

- There is (at least) a queen in row 1: $x_{11} \vee x_{12}$
- There is (at least) a queen in row 2: $x_{21} \vee x_{22}$
- There are no two queens in the same row:
  $\neg(x_{11} \wedge x_{12}) \wedge \neg(x_{21} \wedge x_{22})$
  that is in CNF: $(\neg x_{11} \vee \neg x_{12}) \wedge (\neg x_{21} \vee \neg x_{22})$
- There are no two queens in the same column:
  $(\neg x_{11} \vee \neg x_{21}) \wedge (\neg x_{12} \vee \neg x_{22})$
- There are no two queens in the same diagonal:

Encoding the 2-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 2$).

Constraints:

- There is (at least) a queen in row 1: $x_{11} \vee x_{12}$
- There is (at least) a queen in row 2: $x_{21} \vee x_{22}$
- There are no two queens in the same row:
  $\neg(x_{11} \wedge x_{12}) \wedge \neg(x_{21} \wedge x_{22})$
  that is in CNF: $(\neg x_{11} \vee \neg x_{12}) \wedge (\neg x_{21} \vee \neg x_{22})$
- There are no two queens in the same column:
  $(\neg x_{11} \vee \neg x_{21}) \wedge (\neg x_{12} \vee \neg x_{22})$
- There are no two queens in the same diagonal:
  $(\neg x_{11} \vee \neg x_{22}) \wedge (\neg x_{12} \vee \neg x_{21})$

One could also add constraints expressing that there is (at least) a queen in each column. These are optional: they are implied by the constraints above.

Encoding the 3-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 3$).

Encoding the 3-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 3$).

Constraints:

- There is (at least) a queen in row 1:

Encoding the 3-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 3$).

Constraints:

- There is (at least) a queen in row 1: $x_{11} \vee x_{12} \vee x_{13}$
- Likewise for rows 2 and 3.
- There are no two queens in row 1:

Encoding the 3-Queens Problem in SAT

Propositional variables:
Let $x_{r,c}$ mean "there is a queen in row r and column c" (for $1 \leq r, c \leq 3$).

Constraints:

- There is (at least) a queen in row 1: $x_{11} \vee x_{12} \vee x_{13}$
- Likewise for rows 2 and 3.
- There are no two queens in row 1:
  $(\neg x_{11} \vee \neg x_{12}) \wedge (\neg x_{11} \vee \neg x_{13}) \wedge (\neg x_{12} \vee \neg x_{13})$
- Likewise for rows 2 and 3, all three columns, and all diagonals.

Exercise: Generalise this encoding to the n-queens problem.