# Exam in Algorithms & Data Structures 3 (1DL481)

Prepared by Pierre Flener

Wednesday 13 March 2024 from 14:00 to 17:00 in hall 1 at Bergsbrunnagatan 15

**Materials:** This is a ***closed***-book exam, drawing from the book *Introduction to Algorithms* by T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, published in 4th edition by The MIT Press in 2022, and denoted by CLRS4 here. ***No*** means of help are allowed.

**Instructions:** Question 1 is ***mandatory***: you must earn ***at least half*** of its points in order to pass this exam (see **Grading** below). Start each answer on a new sheet, and indicate there the question number. Your answers must be written in English. Unreadable, unintelligible, and irrelevant answers will not be considered. Provide only the requested information and nothing else, but always show ***all*** the details of your reasoning, unless explicitly not requested, and make explicit ***all*** your additional assumptions. Do ***not*** write anything into the following table:

| Question | Max Points | Your Mark |
|---|---|---|
| 1 | 7 | |
| 2 | 4 | |
| 3 | 4 | |
| 4 | 5 | |
| Total | 20 | |

**Help:** No teacher will attend the exam. If you think a question is unclear, then explain your difficulty with the question and state the assumption that underlies your answer.

**Grading:** Your grade is as follows when your exam mark is $e$ points, including ***at least*** 3.5 points on Question 1:

| Grade | Condition |
|---|---|
| 5 | $18 \leq e \leq 20$ |
| 4 | $14 \leq e \leq 17$ |
| 3 | $10 \leq e \leq 13$ |
| U | $00 \leq e \leq 09$ |

**Identity:** Your anonymous exam code:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 7 | F | 2 | B | - | 0 | 0 | | | - | | | |

## Question 1: NP-Completeness (*mandatory question*!) (7 points)

Exercise 34.5-2: Given an integer $m \times n$ matrix $A$ and an integer $m$-vector $b$, the ***0-1 integer-programming problem*** asks whether there exists an integer $n$-vector $x$ with elements in the set $\{0, 1\}$ such that $A\,x \leq b$. Prove that 0-1 integer programming is NP-complete, by using a *single* reduction, *directly* from ***3-CNF satisfiability***, which asks whether a conjunction of clauses, each of exactly three distinct literals, is satisfiable; a *literal* is an occurrence of a Boolean variable or its negation.

Start your answer on a new sheet. You must earn ***at least half*** of the points of this question. You ***must*** use the identifiers $A$, $m$, $n$, $b$, and $x$ of the question.

## Question 2: Probabilistic Analysis (4 points)

Exercise 5.2-5: Use indicator random variables to solve the following problem, which is known as the ***hat-check problem***. Each of $n$ customers gives a hat to a hat-check person at a restaurant. The hat-check person gives the hats back to the customers in a random order. What is the expected number of customers who get back their own hat? Are your indicator random variables independent?

Start your answer on a new sheet.

## Question 3: Amortised Analysis (4 points)

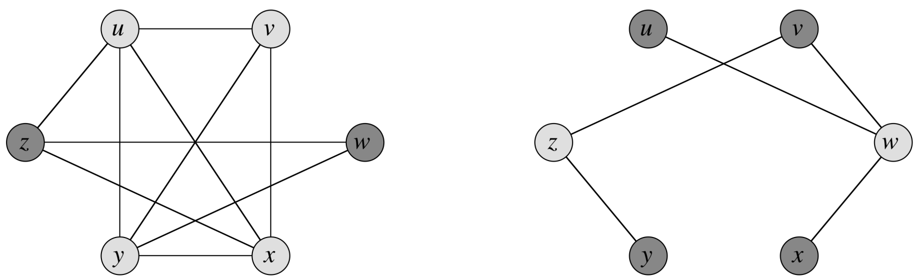
Exercise 16.2-2: Suppose we perform a sequence of $n$ operations on a data structure in which the $i$th operation costs $i$ if $i$ is an exact power of 2, and 1 otherwise. Use an *accounting method of analysis* (also known as the *banker's method*) to determine the amortised cost per operation.

Start your answer on a new sheet.

## Question 4: Approximation Algorithms (5 points)

Exercise 35.1-5: [First consider some reminders.] For an undirected graph $G = (V, E)$:

- A ***clique*** in $G$ is a subset $V' \subseteq V$ of vertices, each pair of which is connected by an edge in $E$. The ***size*** of a clique is the number of vertices it contains. The ***clique problem*** is the optimisation problem of finding a clique of maximum size in a graph. For example, a maximum-size clique of the graph in Figure (a) is $\{u, v, x, y\}$, of size 4.
- A ***vertex cover*** of $G$ is a subset $V' \subseteq V$ such that if $(u, v) \in E$, then $u \in V'$ or $v \in V'$ (or both). The ***size*** of a vertex cover is the number of vertices in it. The ***vertex-cover problem*** is the optimisation problem of finding a vertex cover of minimum size in a given graph. For example, a minimum-size vertex cover of the graph in Figure (b) is $\{w, z\}$, of size 2. Note that there is a polynomial-time 2-approximation algorithm for the vertex-cover problem and recall that we saw a polynomial-time 2-approximation algorithm (based on LP rounding) for the *weighted* vertex-cover problem, both analyses being tight.
- The ***complement*** of $G$ is $\overline{G} = (V, \overline{E})$, where $\overline{E} = \{(u, v) : u, v \in V, u \neq v, (u, v) \notin E\}$. For example, the graphs in the following figure are each other's complements:




(a) (b)

[Now comes the exam question itself:] The proof (by reduction from a decision version of the clique problem) of Theorem 34.12, which states that a decision version of the vertex-cover problem is NP-complete, illustrates that the vertex-cover problem and the NP-complete clique problem are complementary in the sense that a minimum-size vertex cover is the complement of a maximum-size clique in the complement graph. For example, the minimum-size vertex cover $\{w, z\}$ of the graph in Figure (b) is the complement $V \setminus V'$ of the maximum-size clique $V' = \{u, v, x, y\}$ in the complement graph in Figure (a). Does this relationship imply that there is a polynomial-time approximation algorithm with a constant approximation ratio for the clique problem? Justify your answer, and first clarify whether you use the CLRS4 convention or the Princeton convention for the approximation of maximisation problems.

Start your answer on a new sheet. You ***must*** use the identifiers and notation $G$, $V$, $E$, $V'$, $\overline{G}$, and $\overline{E}$ of the question.