Fuzzy version of Sum of minimal distances

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Outline

- Current work:
  - Sum of minimal distances for crisp (binary) images
- Project work:
  - Sum of minimal distances for fuzzy (grey-scale) images

Problem

- How to extend the Sum of minimal distances to be a useful distance measure for grey-scale images?

$$d_{SMD}(A, B) = \frac{1}{2}\Big(\sum_{a \in A} d(a, B) + \sum_{b \in B} d(b, A) \Big)$$

- Desirable properties for a new distance d are:
    - Positivity: $d(A, B) \geq 0$
    - Reflexivity: $d(A, A) = 0$
    - Separability: $d(A, B) = 0 \Leftrightarrow A = B$
    - Symmetry: $d(A, B) = d(B, A)$
    - Triangular inequality: $d(A, B) \leq d(A, C) + d(C, B)$

Different possibilities how to extend the crisp distance to fuzzy distance

- Consider fuzzy set in n dimensional space as $n+1$ dimensional crisp set
- Fuzzification principle
- Weighting distances by membership function
- Fuzzy distances as a fuzzy set instead of as a crisp number

Fuzzification principle

- Let d_{SMD} be the distance between crisp sets, then its fuzzy equivalent is defined by
  - $$d_{SMD}(A, B) = \int_0^1 d_{SMD}({}^{\alpha}A, {}^{\alpha}B) d\alpha$$
  - $$d_{SMD}(A, B) = \sup_{\alpha > 0} d_{SMD}({}^{\alpha}A, {}^{\alpha}B)$$

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Problem: $height(A) \neq height(B)$.

Attacking the problem of different heights

$$d_{SMD}(A,B) = \int_0^1 w(\alpha) d_{SMD}({}^{\alpha}A', {}^{\alpha}B') d\alpha + \varepsilon \frac{d_1(A,B)}{|X|},$$

where:

- $w(\alpha)$ is any function $\int_0^1 w(\alpha) d\alpha = 1$
- A', B' are normal fuzzy sets such that $A'(x) = A(x)$, where $A(x) < height(A)$ and $A'(x) = A(x)$ otherwise
- d_1 is the L_1-norm
- ε is a small value (ugly) value

Point to set distance

- For the crisp case

$$d(a,B) = \inf_{b \in B} d(a,b).$$

- For the fuzzy case
  - Weighting

$$d(a,B) = \inf_{b \in B} \Big(d(a,b) \cdot f(B(b)) \Big),$$

where $f(t)$ is decreasing function with decreasing t.

  - Fuzzification

$$d(a,B) = \int_0^1 \min_{b \in {}^{\alpha}B} d(a,b) d\alpha$$

Attacking the problem of point to set distance

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- Idea: One distance measure for $x \in Supp(A) \cap Supp(B)$ and another one for $x \notin Supp(A) \cap Supp(B)$

Conclusions

- Fuzzy methods are useful and I did not solve this problem
- Distance between point and fuzzy set is still open question
- Already exist many different approaches for fuzzy distances
- More freedom then in the crisp case