Fuzzy Sets and Fuzzy Techniques

Lecture 9 – Fuzzy numbers and fuzzy arithmetics

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Topics of today

- Interval numbers
- Interval arithmetics
- Fuzzy numbers
- Fuzzy arithmetics
- A lattice of fuzzy numbers, MIN and MAX operators
- Fuzzy equations
- An example of fuzzy linear programming

Interval numbers

An **interval number**, representing an **uncertain** real number

$$A = [a_1, a_2] = \{x \mid a_1 \leq x \leq a_2, x \in \mathbb{R}\}$$

A real number is a special case: **point interval** or **singleton**

$$a = [a, a]$$

Interval numbers

For the interval A we define

Width

$$w(A) = a_2 - a_1$$

Magnitude

$$|A| = \max(|a_1|, |a_2|)$$

(not to confuse with cardinality)

Image

$$A^- = [-a_2, -a_1]$$

Inverse

$$A^{-1} = [\frac{1}{a_2}, \frac{1}{a_1}]$$

Interval numbers

Equality

$$A = B \;\Leftrightarrow\; a_1 = b_1 \text{ and } a_2 = b_2$$

Inclusion

$$A \subseteq B \;\Leftrightarrow\; b_1 \leq a_1 \leq a_2 \leq b_2$$

Distance

$$d(A, B) = \max(|a_1 - b_1|, |a_2 - b_2|)$$

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Fuzzy numbers and fuzzy intervals
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Linear programming
Fuzzy linear programming

Arithmetic operations on intervals

For intervals A and B, and operator $* \in \{+, -, \cdot, /\}$ we define

$$A * B = \{a * b \mid a \in A, b \in B\}$$

Division, A/B, is not defined when $0 \in B$.

The result of an arithmetic operation on closed intervals is again a closed interval.

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Arithmetic operations on intervals

For closed intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$, the four arithmetic operations are defined as follows (equivalent with definition on previous slide)

$$\begin{aligned}
A + B \quad &= [a_1 + b_1, a_2 + b_2] \\
A - B = A + B^- \quad &= [a_1 - b_2, a_2 - b_1] \\
A \cdot B \quad &= [\min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2), \\
&\quad\; \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)]
\end{aligned}$$

and, if $0 \notin [b_1, b_2]$

$$\begin{aligned}
A/B = A \cdot B^{-1} \quad &= [a_1, a_2] \cdot [\tfrac{1}{b_2}, \tfrac{1}{b_1}] \\
&= [\min(\tfrac{a_1}{b_1}, \tfrac{a_1}{b_2}, \tfrac{a_2}{b_1}, \tfrac{a_2}{b_2}), \max(\tfrac{a_1}{b_1}, \tfrac{a_1}{b_2}, \tfrac{a_2}{b_1}, \tfrac{a_2}{b_2})].
\end{aligned}$$

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Arithmetic operations on intervals

Let $\mathbf{0} = [0, 0]$ and $\mathbf{1} = [1, 1]$

1. $A + B = B + A$,
   $A \cdot B = B \cdot A$ (commutativity)
2. $(A + B) + C = A + (B + C)$,
   $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ (associativity)
3. $A = \mathbf{0} + A = A + \mathbf{0}$
   $A = \mathbf{1} \cdot A = A \cdot \mathbf{1}$ (identity)
4. $A \cdot (B + C) \subseteq A \cdot B + A \cdot C$ (subdistributivity)
5. If $b \cdot c \geq 0$ for every $b \in B$ and $c \in C$, then
   $A \cdot (B + C) = A \cdot B + A \cdot C$ (distributivity)
   Furthermore, if $A = [a, a]$, then $a \cdot (B + C) = a \cdot B + a \cdot C$
6. $0 \in A - A$ and $1 \in A/A$
7. If $A \subseteq E$ and $B \subseteq F$, then:
   $$\begin{aligned} A + B &\subseteq E + F \\ A - B &\subseteq E - F \\ A \cdot B &\subseteq E \cdot F \\ A/B &\subseteq E/F \text{ (inclusion monotonicity)} \end{aligned}$$

Arithmetic operations on intervals

Note:

(i) $A - A = A + A^{-} = [-(a_2 - a_1), a_2 - a_1] \neq [0, 0]$

(ii) $A/A = A \cdot A^{-1} \neq [1, 1]$

However (from prev. slide)

6. $0 \in A - A$ and $1 \in A/A$

Multi-level interval numbers

Remark:
A path to fuzzy numbers (and then further on to fuzzy sets) taken by Bojadziev and Bojadziev is via two- and multi-level interval numbers, where different parts of the interval have different *plausibility*.

We, however, already know about fuzzy sets...

Fuzzy numbers and fuzzy intervals

A **fuzzy number** is a fuzzy set on $\mathbb{R}$

$$A : \mathbb{R} \to [0, 1]$$

such that

(i) A is normal ($\text{height}(A) = 1$)

(ii) ${}^{\alpha}A$ is a *closed* interval for all $\alpha \in (0, 1]$

(iii) The support of A, $\text{Supp}(A) = {}^{0+}A$, is bounded

Since all α-cuts are closed intervals, every fuzzy number is a **convex** fuzzy set.

The converse is not necessarily true.

Fuzzy numbers and fuzzy intervals

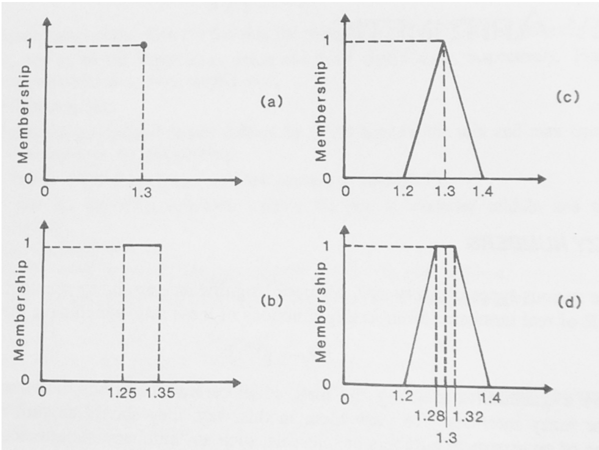

Figure: A comparison of a real number and a crisp interval with a fuzzy number and a fuzzy interval, respectively.

Fuzzy numbers and fuzzy intervals

Theorem (4.1)

Let $A \in \mathcal{F}(\mathbb{R})$. Then, A is a fuzzy number iff there exists a closed interval $[a, b] \neq \emptyset$ such that

$$A(x) = \begin{cases} 1 & \text{for } x \in [a, b] \\ l(x) & \text{for } x \in (-\infty, a) \\ r(x) & \text{for } x \in (b, \infty) \end{cases}$$

where $l : (-\infty, a) \to [0, 1]$ is monotonic non-decreasing, continuous from the right, and $l(x) = 0$ for $x < \omega_1$ and $r : (b, \infty) \to [0, 1]$ is monotonic non-increasing, continuous from the left, and $r(x) = 0$ for $x > \omega_2$.

Fuzzy numbers and fuzzy intervals

Fuzzy cardinality

An example of a function that maps to the set of fuzzy numbers.

Given a finite fuzzy set A, its **fuzzy cardinality** $|\tilde{A}|$ is a fuzzy number defined on $\mathbb{N}$ by

$$|\tilde{A}|(|^{\alpha}A|) = \alpha$$

Linguistic variables

When fuzzy numbers are connected to linguistic concepts, such as *very small*, *small*, *medium*, and so on, and interpreted in a particular *context*, the resulting constructs are usually called **linguistic variables**.

A linguistic variable is fully characterized by a quintuple $\langle v, T, X, g, m\rangle$, in which v is the **name** of the variable, T is the set of **linguistic terms** of v that refer to the base variable whose values range over a **universal set** X, g is a **syntactic rule** (a grammar) for generating linguistic terms, and m is a **semantic rule** that assigns to each linguistic term $t \in T$ its meaning, $m(t)$, which is a fuzzy set on X (i.e., $m : T \to \mathcal{F}(X)$).

Linguistic variables

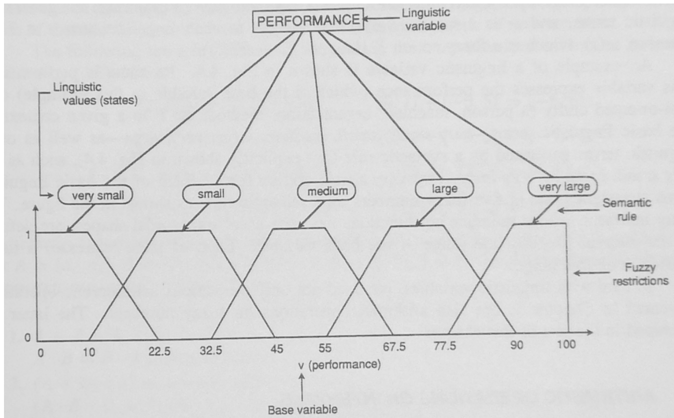

Figure: An example of a linguistic variable.

Arithmetics on fuzzy numbers

Moving from interval numbers, we can define arithmetics on fuzzy numbers based on two principles:

1. **Cutworthiness** (thanks to inclusion monotonicity of intervals)
$$^{\alpha}(A * B) = {}^{\alpha}A * {}^{\alpha}B$$
in combination with
$$A * B = \bigcup_{\alpha \in (0,1]} {}_{\alpha}(A * B)$$

2. or the **extension principle**
$$(A * B)(z) = \sup_{z = x * y} \min\,[A(x), B(y)]$$

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Arithmetics on fuzzy numbers

Theorem (4.2)

Let $* \in \{+, -, \cdot, /\}$, and let A, B denote continuous fuzzy numbers. Then, the fuzzy set $A * B$ defined by the extension principle (prev. slide) is a continuous fuzzy number.

Lemma

$$(A * B)(z) = \sup_{z = x * y} \min\,[A(x), B(y)] \;\Rightarrow\; {}^{\alpha}(A * B) = {}^{\alpha}A * {}^{\alpha}B$$

So the **two definitions are equivalent** for **continuous** fuzzy numbers. (The proof is built on continuity.)

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Arithmetics on fuzzy numbers

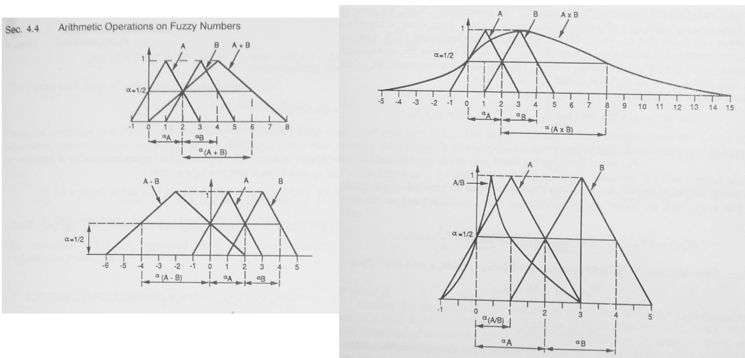

Figure: Illustration of arithmetic operations on fuzzy numbers.

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MIN and MAX operators

$$\mathrm{MIN}(A, B)(z) = \sup_{z = \min(x,y)} \mathit{min}\,[A(x), B(y)],$$
$$\mathrm{MAX}(A, B)(z) = \sup_{z = \max(x,y)} \mathit{min}\,[A(x), B(y)]$$

Again, for continuous fuzzy numbers, this is equivalent with a definition based on cutworthiness.

$$^{\alpha}(\mathrm{MIN}(A, B)) = \mathrm{MIN}({}^{\alpha}A, {}^{\alpha}B),$$
$$^{\alpha}(\mathrm{MAX}(A, B)) = \mathrm{MAX}({}^{\alpha}A, {}^{\alpha}B), \quad \forall \alpha \in (0, 1].$$

Where, for intervals $[a_1, a_2]$, $[b_1, b_2]$
$$\mathrm{MIN}([a_1, a_2], [b_1, b_2]) = [\min(a_1, b_1), \min(a_2, b_2)],$$
$$\mathrm{MAX}([a_1, a_2], [b_1, b_2]) = [\max(a_1, b_1), \max(a_2, b_2)].$$

MIN and MAX operators

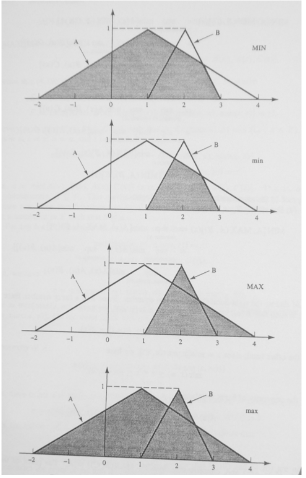

Figure: Comparison of the operators MIN, min, MAX, and max.

MIN and MAX operators

Theorem (4.3)

- $\mathrm{MIN}(A,B) = \mathrm{MIN}(B,A)$,
  $\mathrm{MAX}(A,B) = \mathrm{MAX}(B,A)$ *(commutativity)*
- $\mathrm{MIN}[\mathrm{MIN}(A,B),C] = \mathrm{MIN}[A,\mathrm{MIN}(B,C)]$,
  $\mathrm{MAX}[\mathrm{MAX}(A,B),C] = \mathrm{MAX}[A,\mathrm{MAX}(B,C)]$ *(associativity)*
- $\mathrm{MIN}(A,A) = A$,
  $\mathrm{MAX}(A,A) = A$ *(idempotence)*
- $\mathrm{MIN}[A,\mathrm{MAX}(A,B)] = A$,
  $\mathrm{MAX}[A,\mathrm{MIN}(A,B)] = A$ *(absorption)*
- $\mathrm{MIN}[A,\mathrm{MAX}(B,C)] = \mathrm{MAX}[\mathrm{MIN}(A,B),\mathrm{MIN}(A,C)]$,
  $\mathrm{MAX}[A,\mathrm{MIN}(B,C)] = \mathrm{MIN}[\mathrm{MAX}(A,B),\mathrm{MAX}(A,C)]$
  (distributivity)

Lattice of fuzzy numbers

Denoting the set of fuzzy numbers with $\mathcal{R}$,

$\langle \mathcal{R}, \mathrm{MIN}, \mathrm{MAX}\rangle$ is a **distributive lattice**, in which MIN and MAX represent the meet and join, respectively.

Recall: A lattice is a **partially ordered set** (or poset) whose nonempty finite subsets all have a unique supremum (called join) and an infimum (called meet).

The lattice can also be expressed $\langle \mathcal{R}, \preceq\rangle$, where $\preceq$ is a **partial ordering** on $\mathcal{R}$ given by

$$A \preceq B \;\Leftrightarrow\; \mathrm{MIN}(A,B) = A \;\text{ or, alternatively}$$
$$A \preceq B \;\Leftrightarrow\; \mathrm{MAX}(A,B) = B$$

Lattice of fuzzy numbers

The partial order expressed in terms of α-cuts

$$A \preceq B \;\Leftrightarrow\; \mathrm{MIN}({}^{\alpha}A, {}^{\alpha}B) = {}^{\alpha}A,$$
$$A \preceq B \;\Leftrightarrow\; \mathrm{MAX}({}^{\alpha}A, {}^{\alpha}B) = {}^{\alpha}B,$$

for all $\alpha \in (0,1]$.
Where, for intervals $[a_1,a_2]$, $[b_1,b_2]$

$$[a_1,a_2] \le [b_1,b_2] \;\Leftrightarrow\; a_1 \le b_1 \text{ and } a_2 \le b_2$$

Thus, for any $A, B \in \mathcal{R}$, we have

$$A \preceq B \text{ iff } {}^{\alpha}A \le {}^{\alpha}B$$

for all $\alpha \in (0,1]$.

Lattice of fuzzy numbers

Not all fuzzy numbers are comparable (only partial order).
However, values of linguistic variables are often defined by fuzzy numbers that are comparable.
For example:

$$\text{very small} \preceq \text{small} \preceq \text{medium} \preceq \text{large} \preceq \text{very large}$$

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Interval equations

Equation $A + X = B$

Equation $A + X = B$
$X = B - A$ is **not** the solution.

Take two closed intervals $A = [a_1, a_2], B = [b_1, b_2]$
Then, $B - A = [b_1 - a_2, b_2 - a_1]$ and

$$\begin{aligned} A + (B - A) &= [a_1, a_2] + [b_1 - a_2, b_2 - a_1] \\ &= [a_1 + b_2 - a_2, a_2 + b_2 - a_1] \\ &\neq [b_1, b_2] = B, \end{aligned}$$

whenever $a_1 \neq a_2$.

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Interval equations

Equation $A + X = B$

$A + X = B$
Let $X = [x_1, x_2]$.
Then $[a_1 + x_1, a_2 + x_2] = [b_1, b_2]$ follows immediately.
Clearly: $x_1 = b_1 - a_1$ and $x_2 = b_2 - a_2$.
Since X must be an interval, it is required that $x_1 \leq x_2$.
That is, the equation has a solution iff $b_1 - a_1 \leq b_2 - a_2$.
Then $X = [b_1 - a_1, b_2 - a_2]$ is the solution.

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Fuzzy equations

The solution to a fuzzy equation can be obtained by solving a set of interval equations, one for each nonzero α in the level set $\Lambda(A) \cup \Lambda(B)$.

The equation $A + X = B$ has a solution iff

(i) ${}^\alpha b_1 - {}^\alpha a_1 \leq {}^\alpha b_2 - {}^\alpha a_2$ for every $\alpha \in (0, 1]$, and
(ii) $\alpha \leq \beta$ implies
${}^\alpha b_1 - {}^\alpha a_1 \leq {}^\beta b_1 - {}^\beta a_1 \leq {}^\beta b_2 - {}^\beta a_2 \leq {}^\alpha b_2 - {}^\alpha a_2$.

If a solution ${}^\alpha X$ exists for every $\alpha \in (0, 1]$ (property (i)), and property (ii) is satisfied, then the solution X is given by

$$X = \bigcup_{\alpha \in (0,1]} {}_\alpha X$$

Fuzzy equations

Equation $A \cdot X = B$

Similarly as $A + X = B$

The equation $A \cdot X = B$ has a solution iff

(i) ${}^{\alpha}b_1 / {}^{\alpha}a_1 \leq {}^{\alpha}b_2 / {}^{\alpha}a_2$ for every $\alpha \in (0, 1]$, and

(ii) $\alpha \leq \beta$ implies ${}^{\alpha}b_1 / {}^{\alpha}a_1 \leq {}^{\beta}b_1 / {}^{\beta}a_1 \leq {}^{\beta}b_2 / {}^{\beta}a_2 \leq {}^{\alpha}b_2 / {}^{\alpha}a_2$.

If the solution exists, it has the form

$$X = \bigcup_{\alpha \in (0,1]} {}_{\alpha}X$$

where ${}^{\alpha}X = [{}^{\alpha}b_1 / {}^{\alpha}a_1, {}^{\alpha}b_2 / {}^{\alpha}a_2]$.

Again, $X = B/A$ is not a solution of the equation.

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Linear programming

The classical linear programming problem is to find the minimum or maximum values of a linear function under constraints represented by linear inequalities or equations.

The most typical linear programming problem is:

Minimize (or maximize) $c_1x_1 + c_2x_2 + \ldots + c_bx_n$

Subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\leq b_2 \\ &\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\leq b_m \\ x_1, x_2, \ldots, x_n &\geq 0. \end{aligned}$$

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Linear programming

The same thing in vector notation

Minimize $z = \mathbf{cx}$

Subject to $\mathbf{Ax} \leq \mathbf{b}$

$\mathbf{x} \geq \mathbf{0}$

Where:

- the function to be minimized, z, is called an **objective function**
- the vector $\mathbf{c} = (c_1, \ldots, c_n)$ is called a **cost vector**
- the matrix $\mathbf{A} = [a_{ij}]$ is called a **constraint matrix**
- the vector $\mathbf{b} = (b_1, \ldots, b_n)^T$ is called a **right-hand-side vector**
- and the vector $\mathbf{x} = (x_1, \ldots, x_n)^T$ is our vector of variables.

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Linear programming

An example

Minimize $z = x_1 - 2x_2$

Subject to

$$\begin{aligned} 3x_1 - x_2 &\geq 1 \\ 2x_1 + x_2 &\leq 6 \\ 0 \leq x_2 &\leq 2 \\ 0 &\leq x_1 \end{aligned}$$

The **feasible set**, i.e., the set of vectors $\mathbf{x}$ that satisfy all constraints, is *always a* **convex polygon** (if bounded).

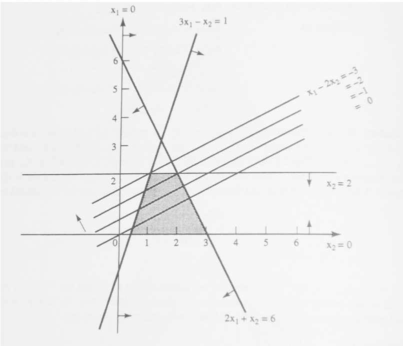

Figure 15.6 An example of a classical linear programming problem.

Figure: An example of a classical linear programming problem.

Fuzzy linear programming

In many practical situations, it is not reasonable to require that the constraints or the objective function are specified in crisp precise terms.

The most general case of fuzzy linear programming grows rather complex, and is not discussed in the book.

A realistic special case to provide the feeling:

The situation where the right-hand-side vector and the constraint matrix are expressed by fuzzy triangular numbers.

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Fuzzy linear programming

$$\text{Minimize} \quad z = \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} A_{ij} x_j \leq B_i \quad , \ i = 1 \ldots m$$

$$x_j \geq 0 \quad , \ j = 1 \ldots n$$

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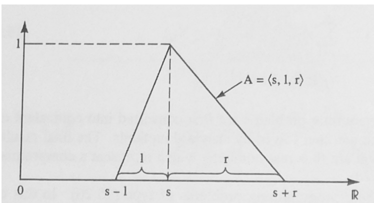
Fuzzy linear programming

Any triangular fuzzy number can be represented by three real numbers, s, l, r.



(b) Triangular fuzzy numbers employed in (15.21).

Figure: Triangular fuzzy number, represented by the triple $\langle s, l, r \rangle$.

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Fuzzy linear programming

$$\text{Minimize} \quad z = \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} \langle s_{ij}, l_{ij}, r_{ij} \rangle x_j \leq \langle t_i, u_i, v_i \rangle \quad , \ i = 1 \ldots m$$

$$x_j \geq 0 \quad , \ j = 1 \ldots n$$

Where $A_{ij} = \langle s_{ij}, l_{ij}, r_{ij} \rangle$, $B_i = \langle t_i, u_i, v_i \rangle$ and $\leq$ is defined by the partial order $\preceq$, i.e., $A \leq B$ iff $\mathrm{MAX}(A, B) = B$.

Fuzzy linear programming

For any two triangular numbers, $A = \langle s_1, l_1, r_1 \rangle$ and $B = \langle s_2, l_2, r_2 \rangle$,

$$A \leq B \;\text{ iff }\; \begin{cases} s_1 \leq s_2, & \text{and} \\ s_1 - l_1 \leq s_2 - l_2 & \text{and} \\ s_1 + r_1 \leq s_2 + r_2 & \end{cases}$$

Moreover $\langle s_1, l_1, r_1 \rangle + \langle s_2, l_2, r_2 \rangle = \langle s_1 + s_2, l_1 + l_2, r_1 + r_2 \rangle$ and $\langle s_1, l_1, r_1 \rangle x = \langle s_1 x, l_1 x, r_1 x \rangle$ for any non-negative real number x.

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Fuzzy linear programming

This enables us to rewrite the problem as follows:

$$\begin{array}{ll} \text{Minimize} & z = \sum_{j=1}^{n} c_j x_j \\ \text{Subject to} & \sum_{j=1}^{n} s_{ij} x_j \leq t_i \quad , \; i = 1 \ldots m \\ & \sum_{j=1}^{n} (s_{ij} - l_{ij}) x_j \leq t_i - u_i \quad , \; i = 1 \ldots m \\ & \sum_{j=1}^{n} (s_{ij} + r_{ij}) x_j \leq t_i + v_i \quad , \; i = 1 \ldots m \\ & x_j \geq 0 \quad , \; j = 1 \ldots n \end{array}$$

which is a classic (crisp) linear programming problem that we know how to solve :-)