Fuzzy Sets and Fuzzy Techniques

Lecture 6 – Features of Fuzzy Sets

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Topics of today

- Spatial fuzzy sets
- Scalar descriptors of (spatial) fuzzy sets
  - Definitions
  - Inter-relations
- Other shape descriptors
  - Vector-valued and non-numerical
- Feature estimation
  - Area and higher order moments
  - Perimeter

Spatial fuzzy sets

A couple of properties and features of a fuzzy set A are already mentioned:

- Height, Width
  Largest membership value reached.
  Integral of a projection in one, or another direction.
- Convexity
  Defined as a cutworthy property, from the α-cuts of A
- Fuzziness
  No crisp counterpart, new and independent definition

Spatial fuzzy sets

N. Sladoje,
"Reviews of Scientific Papers on Fuzzy Shape Analysis",
Internal report No. 32, Centre for Image Analysis, Swedish University of Agricultural Sciences, Uppsala University, 48 pages, 2005.
http://www.cb.uu.se/~natasa/Fuzzy_Shape_Analysis.pdf

Fuzzy sets in image processing

A bit more than forty years ago (1965), fuzzy sets were introduced by Zadeh. 14 years later (1979), Rosenfeld introduced fuzzy sets into image analysis [1]. The results obtained in the first five year period (1979-1984), are reported in (Rosenfeld 1984) [2]; various definitions, methods for measuring geometrical and other properties and relationships related to regions in an image defined as fuzzy sets, are summarized.

[1] A. Rosenfeld., Fuzzy digital topology. Information and Control, 1979

[2] A. Rosenfeld., The fuzzy geometry of image subsets. Pat. Rec. Letters, 1984.

Spatial fuzzy sets

The fuzzy shape analysis techniques addressed in (Rosenfeld 1984) are:

- Connectedness and surroundedness;
- Adjacency;
- Convexity and starshapedness;
- Area, perimeter, and compactness;
- Extent and diameter;
- Shrinking and expanding, medial axes, elongatedness, and thinning;
- Grey-level-dependent properties; splitting and merging.

Spatial fuzzy sets

The fact that (Rosenfeld 1984) is, during the last twenty years, still one of the main references in most of the papers dealing with fuzzy shapes, indicates not only its outstanding significance and quality, but also the lack of research and results in the field since then.

However, during the last ten years, there has been a steady increase in interest for fuzzy techniques in image processing...

Geometry and shape of spatial fuzzy sets

Basic shapes

A number of definitions for basic shapes have been proposed.

- Fuzzy point
- Fuzzy line/curve
- Fuzzy disk
- Fuzzy ellipse
- Fuzzy rectangle
- ...

Basic fuzzy shapes

Most definitions rely on cutworthiness.

- A **fuzzy straight/curved line** is a fuzzy set for which any α-cut, $\alpha \in (0, 1]$, is either empty, or a straight/curved line.
- A **fuzzy disk** is a fuzzy set whose non-empty α-cuts, for $\alpha \in (0, 1]$, are concentric disks.
- A **fuzzy ellipse** is a fuzzy set whose non-empty α-cuts, for $\alpha \in (0, 1]$, are ellipses with the same center, orientation, and eccentricity.

Shape definitions - Fuzzy disk

A fuzzy disk is defined by Rosenfeld and Haber, 1985, as a fuzzy set for which the membership of a point to the fuzzy disk depends only on the distance of the point to the centre of the disk. Note that this definition does not ensure that a fuzzy disk is a convex fuzzy set.

However, the definition of a fuzzy disk given by Bogomolny, 1987, requires convexity; a fuzzy disk is a *convex* fuzzy set given by a membership function dependent only on the distance of a point to the centre of the disk. We adhere to this later definition, which ensures that all α-cuts of a fuzzy disk are concentric disks.

Definitions

Aggregating over α-cuts

Given a function $f : \mathcal{P}(X) \to \mathbb{R}$.

We can extends this function to $f : \mathcal{F}(X) \to \mathbb{R}$, using one of the following equations

$$f(A) = \int_0^1 f(^{\alpha}A)\, d\alpha, \tag{1}$$

$$f(A) = \sup_{\alpha \in (0,1]} [\alpha f(^{\alpha}A)] \tag{2}$$

Both these definitions provide consistency for the crisp case.

Definitions

Features derived by aggregation over α-cuts

- Area/volume and other geometric moments
- Perimeter/surface area
- Moments (moment invariants)

Area of a fuzzy set

The **area** of a fuzzy set A on $X \subseteq \mathbb{R}$ is

$$\begin{aligned}\text{area}(A) &= \int_X A(x)\,dx \\ &= \int_0^1 \text{area}({}^\alpha A)\,d\alpha\end{aligned}$$

or, for a digital fuzzy set A on $X \subseteq \mathbb{Z}$,
with level set $\Lambda(A) = \{\alpha_1, \alpha_2 ... \alpha_n\}$

$$\begin{aligned}\text{area}(A) = |A| &= \sum_X A(x) \\ &= \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) \cdot \text{area}({}^\alpha A)\,,\end{aligned}$$

where $\alpha_0 = 0$

Perimeter of a fuzzy set

The **perimeter** of a fuzzy set A

$$\text{perim}(A) = \int_0^1 \text{perim}({}^\alpha A)\,d\alpha$$

The **perimeter** of a **fuzzy step set** S given by a piecewise constant membership function μ_S, is defined as

$$\text{perim}(S) = \sum_{\substack{i,j,k \\ i<j}} |\mu_{S_i} - \mu_{S_j}| \cdot |l_{ijk}|,$$

where l_{ijk} is the k^{th} arc along which bounded regions S_i and S_j, defined by (constant-valued) membership functions μ_{S_i} and μ_{S_j}, meet.

Note:

- Requires estimation of the arc-lengths of the crisp sets.
- Only "horizontal" version.

Moments of a fuzzy set

Moments of a fuzzy set are among the first defined fuzzy concepts.

Geometric moments:

- The **moment** $m_{p,q}(A)$ of a fuzzy set A defined on $X \subset \mathbb{R}^2$, is

$$m_{p,q}(A) = \iint_X A(x,y)\, x^p y^q\, dxdy\,.$$

- The moment $\tilde{m}_{p,q}(A)$ of a digital fuzzy set A on $X \subset \mathbb{Z}^2$, is

$$\tilde{m}_{p,q}(A) = \sum_{(i,j)\in X} A(i,j)\, i^p j^q\,.$$

for integers $p, q \geq 0$.
The moment $m_{p,q}(S)$ has the order $p + q$.

Moments of a fuzzy set

Remark: The area of a set is the $m_{0,0}$ moment.

Remark: The **centroid** (centre of gravity) of a set is

$$(x_c, y_c) = \left(\frac{m_{1,0}(A)}{m_{0,0}(A)}, \frac{m_{0,1}(A)}{m_{0,0}(A)}\right)$$

A crisp set is **uniquely defined** by its (possibly infinite) set of (p, q)-moments.

The same set of moment is **not** sufficient for recovering a fuzzy sets.

We lack information in the *membership direction*.

Fuzzy moments of a fuzzy set

The **generalized moment** [3] $m_{p,q;\varphi}(A)$ of a fuzzy set A on $X \subset \mathbb{R}^2$, is

$$m_{p,q;\varphi}(A) = \iint\limits_X \left(\int\limits_0^{A(x,y)} \alpha^\varphi d\alpha \right) x^p y^q \, dxdy \ ,$$

for integers $p, q, \varphi \geq 0$.

The generalized moment $m_{p,q;\varphi}(A)$ has the order $p + q + \varphi$.

The generalized moments $m_{p,q;0}(A)$ of a fuzzy shape are equal to the ordinary moments $m_{p,q}(A)$ of the same fuzzy shape A.

[3] Representation and reconstruction of fuzzy disks by moments, Sladoje and Lindblad, Fuzzy Sets and Systems, 158(5), 2007, pp. 517–534

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Inter-relations

All the definitions listed above reduce to the corresponding customary definitions for crisp sets. However, some inter-relations which these notions satisfy in the crisp case, do not hold for the generalized (fuzzified) definitions.

For example: The isoperimetric inequality,

$$4\pi \cdot \mathrm{area}(A) \leq \mathrm{perim}^2(A),$$

gives us a way to define a measure of **compactness** of a set A

$$\mathrm{comp}(A) = \frac{4\pi \cdot \mathrm{area}(A)}{\mathrm{perim}^2(A)},$$

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Outline
Spatial fuzzy sets
Scalar descriptors of fuzzy sets
Definitions
Inter-relations
3D
Other types of descriptors
Estimation of features

Inter-relations

This compactness measure, for crisp sets, is the highest for a disk, for which it is equal to one. All other objects have a lower compactness.

However, for fuzzy sets and definitions given above, the isoperimetric inequality does not hold in general.

In fact, it can be shown that the compactness increases with an increase of fuzziness. A rather unintuitive result in deed.

Inter-relations

Bogomolny proposed (1987) modified definitions, such that they still reduce to their customary crisp counterparts, but the isoperimetric inequality, and also relations between length and area, are fulfilled for a wide class of fuzzy (continuous) sets. However, these definitions are often seen as less intuitive.

For example: The **perimeter** of a fuzzy set S given by a piecewise constant membership function μ_S, is then defined as

$$\text{perim}(S) = \sum_{\substack{i,j,k\\ i<j}} |\sqrt{\mu_{S_i}} - \sqrt{\mu_{S_j}}| \cdot |l_{ijk}|,$$

where l_{ijk} is the k^{th} arc along which bounded regions S_i and S_j, defined by (constant-valued) membership functions μ_{S_i} and μ_{S_j}, meet.

3D

Everything (mentioned so far) generalizes to higher dimensions...

- Volume and higher order moments
- Surface area
- Lengths...

Other types of descriptors

Not only scalar descriptors, but also some vector-valued and non-numerical descriptors are studied for fuzzy sets:

- Signature of a fuzzy set based on the distance from the centroid [4]
- Convexity
- Distances and Distance transforms
- Morphological operations
  - Medial axis transforms and skeletons

[4] J. Chanussot, I. Nyström, N. Sladoje, Shape Signatures of Fuzzy Sets Based on Distance from the Centroid, PRL, 26(6), pp. 735-746, 2005.

Estimation of features

As is well know, features of a continuous spatial shape S, can be estimated from features of its digitization $\mathcal{D}(S)$.

The precision of such estimates is in general limited by the spatial resolution of the digital representation.

Many different types of digitizations exist. Imaging devices often provide an output where the value of each image element (pixel/voxel) reflects on to what extent the element is covered by the image object. This is referred to as **pixel coverage digitization**, or simply area/volume sampling.

For object represented by digital spatial fuzzy sets, where the membership of a point indicates to what extent the pixel/voxel is covered by the imaged continuous (crisp) object, **significant improvements in precision** of feature estimates can be obtained. Especially so, for small objects/limited resolution.

Estimation of features

In theory

Theorem

The generalized moments of a continuous convex fuzzy set with bounded support, whose α-cuts are all 3-smooth convex sets, can be estimated from the generalized moments of its digitization in a grid with spatial resolution r and membership resolution ℓ, by the estimation formula

$$m_{p,q;\varphi}(A) = \frac{1}{r^{p+q+2}\,\ell^{\varphi+1}}\,\tilde{m}_{p,q;\varphi}(\mathcal{D}_\ell(rA)) + \mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right) + \mathcal{O}\left(\frac{1}{\ell}\right),$$

for $p+q \le 2$ and $\varphi \le 1$.

If the set is not 3-smooth (e.g., it has straight edges), the first error term becomes $\mathcal{O}\left(\frac{1}{r}\right)$ instead.

Estimation of features

In theory

Remark: A planar 3-smooth convex set is a convex set in the Euclidean plane whose boundary consists of a finite number of arcs having continuous third order derivatives and a positive curvature at every point, except at the end points of the arcs. These conditions exclude the existence of straight boundary segments.

Estimation of features

In practice

Statistical evaluation of simulated data.

Estimation of features

In practice

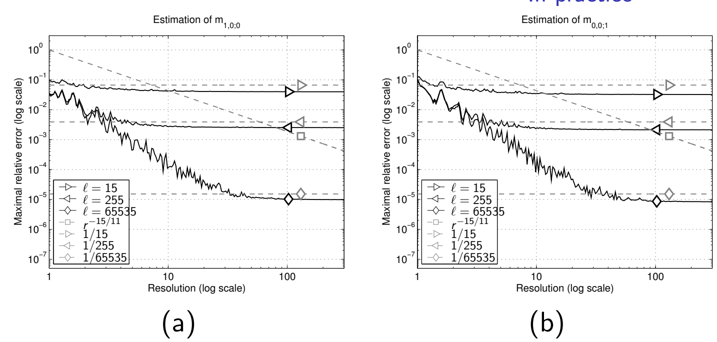


Figure: Maximal relative errors for the estimation of generalized moments and parameters of a continuous fuzzy disk from the generalized moments of its digitization at different spatial and membership resolutions. (a) $m_{1,0;0}$; (b) $m_{0,0;1}$. Dashed lines correspond to asymptotes of the theoretical expressions.

Application - Image registration utilizing moments of a fuzzy shape

Task: Registration of digital X-ray images of hip-prosthesis implants for follow up examinations

Segmentation using active contours, modified to provide pixel coverage values utilized for improved moments' estimation[5] in the registration process [6].

[5] N. Sladoje and J. Lindblad. Estimation of Moments of Digitized Objects with Fuzzy Borders. LNCS-3617, pp. 188-195, 2005.

[6] A. Tanács, C. Domokos, N. Sladoje, J. Lindblad, Z. Kato. Recovering affine deformations of fuzzy shapes. LNCS-5575, pp. 735-744, 2009.

Estimation of features

It is shown that the parameters of the transformation that matches an observation with the template can be obtained as a solution of the following system of equations:

$$|\mathbf{A}| \int_{\mathcal{F}_t} x_k^n \, d\mathbf{x} = \sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{i} \binom{i}{j} q_{k1}^{n-i} q_{k2}^{i-j} q_{k3}^{j} \int_{\mathcal{F}_o} y_1^{n-i} y_2^{i-j} \, d\mathbf{y},$$

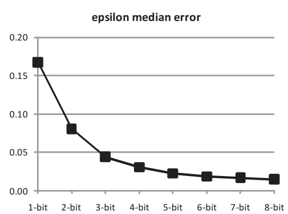
where $k = 1, 2$; $n = 1, 2, 3$ and q_{ki} denote the unknown elements of the inverse transformation $\mathbf{A}^{-1}$.

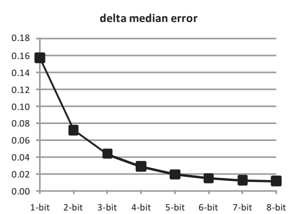
Discretization leads to the following

$$|\mathbf{A}| \sum_{\mathbf{p} \in X_t} \mu_{F_t}(\mathbf{p}) p_k^n = \sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{i} \binom{i}{j} q_{k1}^{n-i} q_{k2}^{i-j} q_{k3}^{j} \sum_{\mathbf{p} \in X_o} \mu_{F_o}(\mathbf{p}) p_1^{n-i} p_2^{i-j}.$$

The coefficients of the system are estimated as moments of a fuzzy set, which increases the precision of the estimation.

Estimation of features





Registration results of 2000 synthetic images using different quantization levels of the fuzzy boundaries.

Error measures used:

$$\epsilon = \frac{1}{m} \sum_{\mathbf{p} \in T} \|(\mathbf{A} - \widehat{\mathbf{A}})\mathbf{p}\|, \quad \text{and} \quad \delta = \frac{|R \triangle O|}{|R| + |O|},$$

Estimation of features

Practical results supporting Rosenfeld

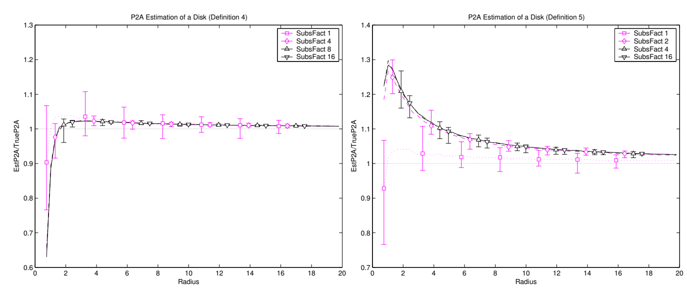


Figure: Inverse compactness measure estimation for digitized disks. Left: Results based on Rosenfelds definitions. Right: Results based on modified definitions (Bogomolny).

A note on the estimation of perimeter

The **perimeter** of a fuzzy set A

$$\text{perim}(A) = \int_0^1 \text{perim}(^{\alpha}A)\, d\alpha$$

The **perimeter** of a **fuzzy step set** S given by a piecewise constant membership function μ_S, is defined as

$$\text{perim}(S) = \sum_{\substack{i,j,k \\ i<j}} |\mu_{S_i} - \mu_{S_j}| \cdot |l_{ijk}|,$$

where l_{ijk} is the k^{th} arc along which bounded regions S_i and S_j, defined by (constant-valued) membership functions μ_{S_i} and μ_{S_j}, meet.

Note: requires estimation of the arc-lengths of the crisp sets.

Estimation of features

Pal and Rosenfeld use the most simple estimate, that is directly counting pixel edges between foreground and background.

A more precise method is described in:

N. Sladoje, I. Nyström, and P.K. Saha, "Measurements of digitized objects with fuzzy borders in 2D and 3D", Image and Vision Computing, vol. 23, pp 123-132, 2005.

Binary case

For a binary configuration there are $2^4 = 16$ possible configurations of pixels forming a Marching Square. Grouping symmetry and complementary cases results in 4 different configurations. The length of the border, corresponding to these configurations, is equal to

- 0, when all 4 pixels are set or when none of them is set;
- $\frac{b}{2}$, when one of the 4 pixels is set or when one of them is not set;
- a, when two edge-neighbours are set;
- b, when two vertex-neighbours are set,

where a and b are the weights assigned to an isothetic and a diagonal step between two neighbouring pixels, respectively. We use $a_{MSE_{n\to\infty}} \approx 0.948$ and $b_{MSE_{n\to\infty}} \approx 1.343$ to minimize the expected mean square error (MSE) for measurement of the length of long line segments ($n \to \infty$) and to give an unbiased estimate. (Note that the weights $a = 1$ and $b = \sqrt{2}$ are not optimal.)

Fuzzy case

The perimeter contribution per of a fuzzy 2×2 configuration c_i to the perimeter of a fuzzy object is

$$per(c_i) = \sum_{j=1}^{4} d_j \cdot b_j\ , \quad \text{for } n = 2\ ,$$

where for $j = 1, 2, 3$, $d_j = \mu_S(x_{j+1}) - \mu_S(x_j)$, $\mu_S(x_j)$ is the membership value of the pixel x_j of c_i, and the memberships within the configuration are sorted in non-decreasing order, i.e., $\mu_S(x_j) \le \mu_S(x_{j+1})$, while b_j is the estimated length of the line corresponding to the binary 2×2 configuration obtained as an α-cut of a fuzzy configuration c_i at level $\alpha = \mu_S(x_{j+1})$.

Fuzzy case

The perimeter of a fuzzy object is obtained as the sum of the contributions of all fuzzy 2×2 configurations contained in the fuzzy segmented image:

$$Per = \sum_i per(c_i) \ .$$

Fuzzy case

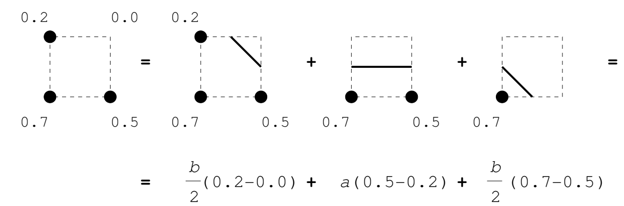


Figure: Calculation of the contribution to perimeter of a fuzzy Marching Square.

Estimation of features

Significant improvement in the precision of feature estimates can be achieved using a fuzzy approach.

Exploiting fuzzy can provide an alternative to increasing the spatial resolution of the image.

The method is extended to 3D and applied to estimation of surface area of an object.

Pixel coverage digitization

If a particular membership function, based on pixel coverage digitization, is used, it is possible to derive a very precise perimeter estimation method.

Restrictions introduced are intuitive for perimeter estimation of crisp continuous objects. Error bounds can be derived and the method can be adjusted to the particular needs.

Generality of the previously described estimation method is lost.

Pixel coverage digitization - a formal definition

Let a square grid in 2D be given. The Voronoi region associated to a grid point $(i, j) \in \mathbb{Z}^2$ is called pixel $p_{(i,j)}$.

Definition (non-quantized case)

For a given continuous object $S \subset \mathbb{R}^2$, inscribed into an integer grid with pixels $p_{(i,j)}$, the *pixel coverage digitization* of S is

$$\mathcal{D}(S) = \left\{ \left((i,j), \frac{A(p_{(i,j)} \cap S)}{A(p_{(i,j)})} \right) \mid (i,j) \in \mathbb{Z}^2 \right\} ,$$

where $A(X)$ denotes the area of a set X.

Digital images $\longrightarrow$ Quantized grey values

Definition (n-level quantized case)

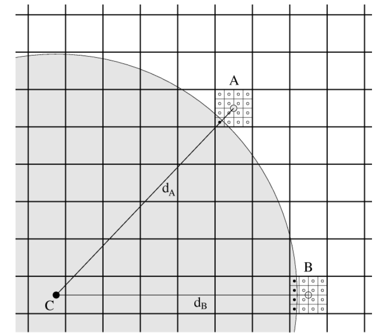
$$\mathcal{D}^n(S) = \left\{ \left((i,j), \frac{1}{n} \left\lfloor n \frac{A(p_{(i,j)} \cap S)}{A(p_{(i,j)})} + \frac{1}{2} \right\rfloor \right) \mid (i,j) \in \mathbb{Z}^2 \right\} ,$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x.

Estimation of features

Remark: Pixel area coverage fuzzification of a disk is not necessarily a digital fuzzy disk.

For the indicated pixels A and B, it holds that $d_A = d(A, C) = d(B, C) = d_B$, while $\mu(A) < \mu(B)$.



Length estimation

An estimator of a straight line segment is derived.

An edge of a halfplane is observed.

Pixel coverage digitization is applied. The length of a straight segment is calculated based on slope estimation.

Examples

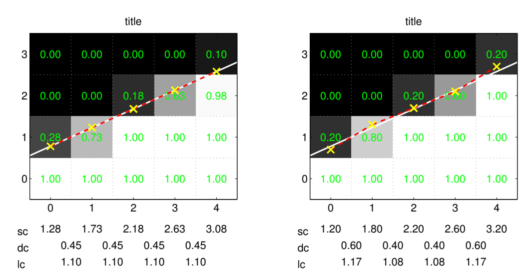


Figure: Example illustrating edge length estimation, l_c, based on difference d_c of column sums s_c for a segment of a halfplane edge given by $y \leq 0.45x + 0.8$ and $n = 5$.

Optimization

Discretization leads to overestimated lengths.

Scaling the estimate with an optimally chosen factor $\gamma_n < 1$, gives an estimate with a minimal error.

$$\hat{l} = \sum_{c=0}^{N-1} \gamma_n \sqrt{1 + d_c^2}$$

The maximal error is minimized for

$$\gamma_n^q = \frac{2q}{q + \sqrt{(\sqrt{n^2+q^2} - n)^2 + q^2}}, \text{ where } q = j - i\ .$$

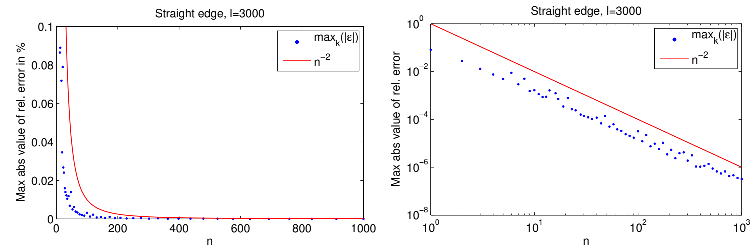
The maximal error is $|\varepsilon| = 1 - \gamma_n^q$[7].
Quantization leads to $q > 1$. In 2D it holds that $q \leq 3$.

[7] N. Sladoje, J. Lindblad, High Precision Boundary Length Estimation by Utilizing Gray-Level Information. IEEE Trans. on Pattern Analysis and Machine Intelligence, Vol. 31, No. 2, pp. 357-363 (2009)

Asymptotic behaviour

Observing the estimation error as a function of the number of grey-levels n, we conclude that

$$|\varepsilon_n| = \mathcal{O}\left(\frac{1}{n^2}\right)\ .$$

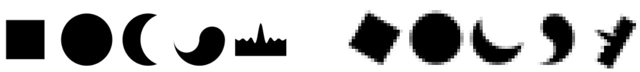


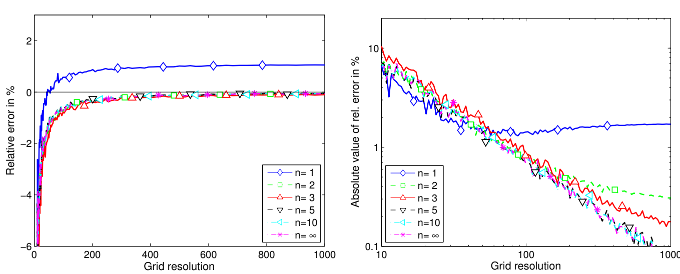
(a) lin-lin scale (b) log-log scale

Figure: Asymptotic behaviour of the maximal error for straight edge length estimation using $\gamma_n = \gamma_n^{(0,1)}$; theoretical (line) and empirical (points) results.

Estimation errors- evaluation of the method

Trade-off between spatial and grey-level resolution





(a) lin-lin scale (b) log-log scale

Figure: Relative errors in percent for test shapes digitized at increasing resolution for 5 different quantization levels and non-quantized ($n = \infty$).

Conclusions

- Spatial fuzzy sets are of a particular interest in image analysis.
- Features of spatial fuzzy sets - shape descriptors.
- "Horizontal" and "vertical" approach in definitions.
- Generalizations are numerous, find the one that suits you.
- High precision estimation - particular membership functions considered.
- Spatial and membership resolution - compensate one for another.
- A number of features are generalized to fuzzy shapes, but still many are left.
- Fuzzy values?