Fuzzy Sets and Fuzzy Techniques

Lecture 13 – Defuzzification

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Outline

Literature

1. W. Van Leekwijck, E.E. Kerre, “Defuzzification: criteria and classification”, *Fuzzy sets and Systems* Vol. 108, pp. 159–178, 1999.
2. T.A. Runkel and M. Glesner, “A set of axioms for defuzzification strategies – towards a theory of rational defuzzification operators”, *Proc. of 2nd IEEE Intern. Conf. on Fuzzy Systems*, pp. 1161–1166, 1993.
3. D. Ralescu, “Average level of a fuzzy set”, *Proc. of IPMU Madrid*, pp. 190–194, 2000
4. E. Roventa, T. Spircu, “Averaging procedures in defuzzification processes”, *Fuzzy sets and Systems*, Vol. 136, pp. 375–385, 2003.
5. L. Rondeau, R. Ruelas, L. Levrat, M. Lamotte, “A defuzzification method respecting the fuzzification”, *Fuzzy Sets and Systems 86*, Vol. 86, 311–320, 1997.

Definition

Definition

Defuzzification is a process that maps a fuzzy set to a crisp set.

Defuzzification has attracted far less attention than other processes involved in fuzzy systems and technologies.

- Defuzzification is sometimes not seen as a part of the core of a fuzzy system; fuzzy system “ends” where uncertainty and imprecision end.
- Defuzzification is “just the last step”; it is an interface with crisp models of the world.
- Defuzzification is a model of synthesis, therefore it is completely opposite to the main concept of fuzzy set theory.

Approaches

- Defuzzification to a point.
- Defuzzification to a set.

- Generating a good representative of a fuzzy set.
- Recovering a crisp original set.

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Criteria

Defuzzification to a point

Classification is made with respect to the mathematical structure of the universal set X.

- **Arbitrary universes**
  (1) Core selection criterion
  (2) Scale invariance (ordinal, interval, ratio, relative, absolute)
- **Ordered universes**
  (3) Monotony
  (4) t-conorm criterion
- **The set of real numbers**
  (5) x-Translation
  (6) x-Scaling
  (7) Continuity

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Criteria

Defuzzification to a set

- Defuzzification should contain the kernel of a fuzzy set.
- Defuzzification should be consistent with affine transformations.
- Defuzzification should preserve a natural order in the set of fuzzy sets.

  If $F \leq G$, then $D(F) \subseteq D(G)$,

  for fuzzy sets F, G and their averagings $D(F)$ and $D(G)$.

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Defuzzification to a point

A list of methods

- Maxima methods and derivatives
  Selection of an element from the core of a fuzzy set as defuzzification value. Main advantage is simplicity.
- Distribution methods and derivatives
  Conversion of the membership function into a probability distribution, and computation of the expected value. Main advantage is continuity property.
- Area methods
  The defuzzification value divides the area under the membership function in two (more or less) equal parts.

Defuzzification to a point

A list of methods – Maxima methods

- Random choice of maxima

$$Prob(D(A) = x_0) = \begin{cases} \frac{1}{|core(A)|} & x_0 \in core(A), \\ 0 & \text{otherwise} \end{cases}$$

- First of maxima, Last of maxima, Middle of maxima

$$\begin{aligned} FOM(A) &= \min core(A) \\ LOM(A) &= \max core(A) \end{aligned}$$

For $MOM(A)$ it holds that

$$|core(A)_{<MOM(A)}| = |core(A)_{>MOM(A)}| \quad \text{if} \quad |core(A)| \quad \text{is odd}$$

$$|core(A)_{<MOM(A)}| = |core(A)_{>MOM(A)}| \pm 1 \quad \text{if} \quad |core(A)| \quad \text{is even.}$$

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A list of methods – Distribution methods (1)

- Centre of gravity (Set of real numbers)

$$COG(A) = \frac{\sum_{x_{min}}^{x_{max}} x \cdot A(x)}{\sum_{x_{min}}^{x_{max}} A(x)}.$$

- Mean of maxima (Set of real numbers)

$$MeOM(A) = \frac{\sum_{x \in core(A)} x}{|core(A)|}.$$

- Basic defuzzification distribution

$$BADD(A, \gamma) = \frac{\sum_{x_{min}}^{x_{max}} x \cdot A(x)^{\gamma}}{\sum_{x_{min}}^{x_{max}} A(x)^{\gamma}},$$

where $\gamma \in [0, \infty)$ reflects the confidence in the system.

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A list of methods – Distribution methods (2)

- Generalized level set defuzzification

$$GLSD(A, \gamma) = \frac{\sum_{i=1}^{N} c_i m_i \gamma^i}{\sum_{i=1}^{N} c_i \gamma^i}$$

where $\gamma \in (0, \infty)$ is the confidence to the system, N is the number of α-cuts, $c_i = |A_{\alpha_i}|$, and m_i is the average value of the ith α-cut.

- Indexed centre of gravity

$$ICOG(A) = \frac{\sum_{x \in A_{\alpha}} x \cdot A(x)}{\sum_{x \in A_{\alpha}} A(x)},$$

where α is a selected threshold below which all membership values are set to zero.

- Fuzzy mean (combines aggregation and defuzzification)

$$FM(A) = \frac{\sum_{i=1}^{N_A} \alpha_i a_i}{\sum_{i=1}^{N_A} \alpha_i},$$

where N_A is the number of fuzzy output sets, α_i is the degree of consistency obtained by inference rules, and a_i is some numerical value associated with the output fuzzy sets A_i (often $a_i = MOM(A_i)$).

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Defuzzification to a point

A list of methods – Area methods

- Centre of area (COA)
  $COA(A)$ is the value that minimizes the expression

$$\left| \sum_{x=\inf(X)}^{COA(A)} A(x) - \sum_{x=COA(A)}^{\sup(X)} A(x) \right|.$$

Defuzzification to a point

Criteria fulfilment

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
FOM/LOM	yes	yes	yes	max	–	–	–
MOM	yes	yes	yes	–	–	–	–
COG	–	–	yes	–	yes	yes	yes
COA	–	–	yes	–	yes	yes	yes

Defuzzification to a set

Averaging procedures

- α-cuts
  chosen at various levels α.
- Average α-cuts
  based on an integration of set-valued function, called Kudo-Aumann integration.

Defuzzification to a set

Averaging procedures

- α-cuts for $\alpha = 0.5$ and $\alpha = 1$ are very often used;
- attempts to chose some α-cut for $\alpha \in [0.5, 1]$ are made;
- attempts to define averaging which is not necessarily an α-cut are made.

Example:

The separating power of a a fuzzy set F is a set $Z \subset supp(F)$, with a compact support and non-zero Lebesgue measure ($\mu(supp(Z)) \neq 0$), such that the function

$$\Theta_F(Z) = \frac{\int_Z F(v)dv}{\mu(Z)} - \frac{\int_{\bar{Z}} F(v)dv}{\mu(\bar{Z})}$$

attains its supremum for it.
$\bar{Z}$ denotes the complement of Z with respect to the support of F.

$SP(F)$ is an averaging procedure.

Defuzzification to a set

Averaging procedures

Example:
Arithmetic mean of two sets:

$$V_1 @ V_2 = \{\frac{1}{2}(v_1 + v_2) \mid v_1 \in V_1, v_2 \in V_2\}.$$

For two averaging procedures $A, B : F \to C$, their arithmetic mean

$$(A@B)(F) = A(F)@B(F)$$

is an averaging procedure. It is usually denoted by $\frac{1}{2}(A + B)$.

For any $\alpha \in [0, 1]$,

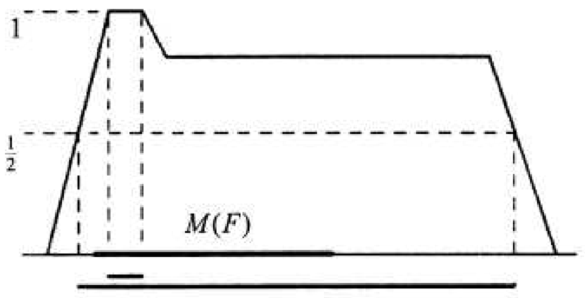
$$A'_\alpha = \frac{1}{2}(A_\alpha + A_{1-\alpha})$$

is an averaging procedure. However, it is not always (intuitively) good.

Defuzzification to a set

Averaging procedures

Possibly better choice is shown below: $M(F) = \frac{1}{2}(A_1 + A_{\frac{1}{2}})$.



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Defuzzification to a set

Average α-cuts

For a set valued function $F : \Omega \to \mathcal{P}(R^n)$, Kudo-Aumann integral is defined as

$$\int_\Omega F\, dm = \left\{ \int_\Omega f\, dm \mid f \in S(F) \right\}$$

where

$$S(F) = \{ f \mid f(w) \in F(w) \ a.e., f \text{ is integrable} \}$$

is the space of integrable **selectors** of F.

Note that the integral of a set-valued function is a set.

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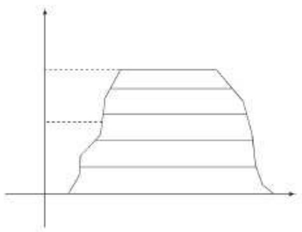
Average α-cuts

Let a fuzzy set A be given by a membership function $\mu : R \to [0,1]$.

- Sets $F(w)$ are α cuts, A_α of the fuzzy set A, for $\alpha \in [0,1]$;
- Selectors are $\varphi(\alpha) = \inf A_\alpha$ and $\phi(\alpha) = \sup A_\alpha$.

Then, the average α-cut of A is

$$A_\mu = \left[\int_{[0,1]} \inf A_\alpha d\alpha, \int_{[0,1]} \sup A_\alpha d\alpha \right].$$



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Average α-cuts

Often, $A_\mu = A_{0.5}$.

Not always.

Example:

$$\mu(x) = \begin{cases} 4(x - x^2), & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

In this case, $A_\mu = [\frac{1}{6}, \frac{5}{6}] = A_{\frac{5}{9}}$.

A_μ is not always an α-cut of A, for any α.

Example:

$$\mu(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 4(x - x^2), & x \in (\frac{1}{2}, 1] \\ 0, & \text{otherwise} \end{cases}$$

In this case, $A_\mu = [\frac{1}{4}, \frac{5}{6}]$. Since $\mu(\frac{1}{4}) \neq \mu(\frac{5}{6})$, $A_\mu \neq A_\alpha$ for every $\alpha \in [0,1]$.

Defuzzification respecting the fuzzification

What is the idea?

Assume:

- universal set being the set of real numbers;
- all parameters of the fuzzification are known;
- a value $y \in R$ is fuzzified and a vector v, characterizing y by fuzzy membership values corresponding to each of the linguistic terms used, is known.

Define defuzzification so that y is recovered from v.

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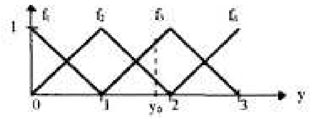
Defuzzification respecting the fuzzification

Example:

Let 4 linguistic terms be represented by fuzzy sets f_i as shown:



Let $y = 1.8$ be given.

$v = (0; 0.2; 0.8; 0)$ is the vector of fuzzy membership values corresponding to y.
Partial solutions of the problem are the sets Y_i, containing elements y_0 such that $f_i(y_0) = v_i$, for each $i = 1, 2, 3, 4$.
In this case, $Y_1 = [1, 3]$, $Y_2 = \{0.2, 1.8\}$, $Y_3 = \{1.8, 2.2\}$, $Y_4 = [0, 2]$.

The solution is defined as the intersection of all partial solutions:
$Y_f = \bigcap_{i=1}^{4} Y_i$
and in this case $Y_f = 1.8$ is obtained.

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Defuzzification respecting the fuzzification

How does it look in reality?

Constraints:

- All the parameters of the fuzzification are known.
- The value v is such that $F(y_0) = v$ for some y_0.

In reality, given v is rarely such, i.e., there is no y_0 fulfilling $F(y_0) = v$.

Instead, the fuzzy representation of a "candidate" for y_0 is generated, using v and fuzzy sets representing linguistic terms (i.e., using knowledge about fuzzification).

To obtain a crisp solution y_0, **defuzzification** (to a point) of its fuzzy representation is performed.

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A bit more general approach

In case when for a given v

$$v \neq F(y), \quad \text{for all } y,$$

a "candidate" for y is found by an optimization (minimization) of the function:

$$J = \frac{1}{2}\sum_{i=1}^{n}(v_i - f_i(y))^2.$$

Defuzzification respecting the fuzzification

A bit more general approach

Assume:

- Functions f_i (used for fuzzification) are triangular. The core of each f_i is a singleton, denoted by C_i.

For a given $v = (v_1, v_2, \ldots, v_n)$, and known $(C_1, C_2, \ldots, C_n)$, we look for y_0 such that

$$y_0 \text{ is equal to } C_i \quad \text{with a weight of } v_i.$$

In other words, y_0 minimizes

$$J = \sum_{i=1}^{n} v_i \cdot D(y_0, C_i) = \sum_{i=1}^{n} v_i (y_0 - C_i)^2.$$

Defuzzification respecting the fuzzification

A bit more general approach

Analytical solution leads to defuzzification by the **height method**, which computes the barycentre of membership function kernels, weighted by the corresponding membership degrees:

$$y_0 = \frac{\sum_{i=1}^{n} v_i C_i}{\sum_{i=1}^{n} v_i}$$

Recall that the fuzzification assumed is based on **triangular** membership functions.

Defuzzification respecting the fuzzification

A bit more general approach - one more step

Constraints on fuzzification:

- For all y, there are at most two f_i (linguistic terms) such that $f_i(y) > 0$.
- The kernel C_i of each f_i is a singleton.
- f_i are monotonically increasing for $x < C_i$ and monotonically decreasing for $x > C_i$.
- The sum of membership values of each element y to all the fuzzy sets f_i is equal to 1.
- All f_i have the same shape (are defined by the same monotonically increasing function f, s.t. $f(0) = 0$ and $f(1) = 1$).

Defuzzification respecting the fuzzification

A bit more general approach - one more step

The aim is to find y_0 such that

$$J = \sum_{i=1}^{n} H(v_i) \cdot D(y_0, C_i) = \sum_{i=1}^{n} H(v_i)(y_0 - C_i)^2,$$

where the function $H : [0,1] \to [0,1]$ appropriately adjusts the weights v_i in the case when non-linear fuzzification is applied, and is, under the assumptions above, shown to be equal to f^{-1}.

The defuzzification method that is in this way developed is

$$y = \frac{\sum_{i=1}^{n} f^{-1}(v_i) C_i}{\sum_{i=1}^{n} f^{-1}(v_i)}.$$

Defuzzification in image processing

Defuzzification by feature distance minimization

Literature:

- N. Sladoje, J. Lindblad, and I. Nyström, “Defuzzification of spatial fuzzy sets by feature invariance”, submitted for publication, Jun 2005.
- J. Lindblad and N. Sladoje, “Feature Based Defuzzification at Increased Spatial Resolution”, Proc. of 11th Intern. Workshop on Combinatorial Image Analysis (IWCIA2006), Berlin, Germany, *Lecture Notes in Computer Science*, Vol. 4040, pp. 131–143, Springer-Verlag, 2006.
- J. Lindblad, N. Sladoje, and T. Lukić, “Feature Based Defuzzification in Z^2 and Z^3 Using a Scale Space Approach”. Proc. of 13th Intern. Conference on Discrete Geometry for Computer Imagery (DGCI2006), Szeged, Hungary, *Lecture Notes in Computer Science*, Vol. 4245, pp. 379–390, Springer-Verlag, 2006.