Fuzzy Sets and Fuzzy Techniques

Lecture 12 – Fuzzy connectedness

Joakim Lindblad, based on a lecture of
László G. Nyúl (2007)

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Outline

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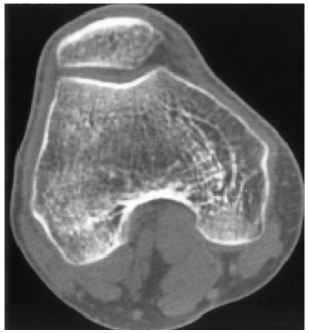
Object characteristics in images



Graded composition

heterogeneity of intensity in the object region due to heterogeneity of object material and blurring caused by the imaging device

Hanging-togetherness

natural grouping of voxels constituting an object a human viewer readily sees in a display of the scene as a Gestalt in spite of intensity heterogeneity

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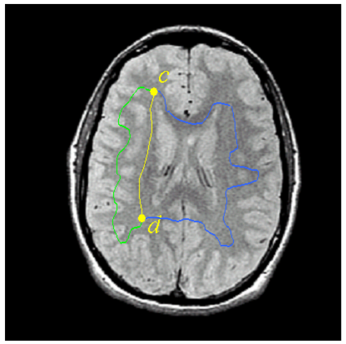
Basic idea of fuzzy connectedness



- local hanging-togetherness (affinity) based on similarity in spatial location as well as in intensity(-derived features)
- global hanging-togetherness (connectedness)

Fuzzy set and relation

A **fuzzy subset** $\mathcal{A}$ of X is

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) \,|\, x \in X\}$$

where $\mu_{\mathcal{A}}$ is the **membership function** of $\mathcal{A}$ in X

$$\mu_{\mathcal{A}} : X \to [0,1]$$

A **fuzzy relation** ρ in X is

$$\rho = \{((x,y), \mu_\rho(x,y)) \,|\, x, y \in X\}$$

with a membership function

$$\mu_\rho : X \times X \to [0,1]$$

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Operations on fuzzy sets

Intersection

$$\mathcal{A} \cap \mathcal{B} = \{(x, \mu_{\mathcal{A}\cup\mathcal{B}}(x)) \,|\, x \in X\} \qquad \mu_{\mathcal{A}\cap\mathcal{B}} = \min(\mu_{\mathcal{A}}, \mu_{\mathcal{B}})$$

Union

$$\mathcal{A} \cup \mathcal{B} = \{(x, \mu_{\mathcal{A}\cup\mathcal{B}}(x)) \,|\, x \in X\} \qquad \mu_{\mathcal{A}\cup\mathcal{B}} = \max(\mu_{\mathcal{A}}, \mu_{\mathcal{B}})$$

Complement

$$\bar{\mathcal{A}} = \{(x, \mu_{\bar{\mathcal{A}}}(x)) \,|\, x \in X\} \qquad \mu_{\bar{\mathcal{A}}} = 1 - \mu_{\mathcal{A}}$$

$\cap$ and $\cup$ are also called T-norm and T-conorm (S-norm).
Several (corresponding pairs) of T- and S-norms exist.
In the FC framework min and max are used.

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Properties of fuzzy relations

ρ is **reflexive** if

$$\forall x \in X \quad \mu_\rho(x,x) = 1$$

ρ is **symmetric** if

$$\forall x, y \in X \quad \mu_\rho(x,y) = \mu_\rho(y,x)$$

ρ is **transitive** if

$$\forall x, z \in X \quad \mu_\rho(x,z) = \bigcup_{y \in X} \mu_\rho(x,y) \cap \mu_\rho(y,z)$$

ρ is **similitude** if it is reflexive, symmetric, and transitive

Note: this corresponds to the equivalence relation in hard sets.

Fuzzy digital space

Fuzzy spel adjacency is a reflexive and symmetric fuzzy relation α in Z^n and assigns a value to a pair of spels (c, d) based on how close they are spatially.

Example

$$\mu_\alpha(c,d) = \begin{cases} \dfrac{1}{\|c-d\|} & \text{if } \|c-d\| < \text{a small distance} \\ 0 & \text{otherwise} \end{cases}$$

Fuzzy digital space

$$(Z^n, \alpha)$$

Scene (over a fuzzy digital space)

$$\mathcal{C} = (C, f) \quad \text{where } C \subset Z^n \text{ and } f : C \to [L, H]$$

Fuzzy spel affinity

Fuzzy spel affinity is a reflexive and symmetric fuzzy relation κ in Z^n and assigns a value to a pair of spels (c, d) based on how close they are spatially and intensity-based-property-wise (local hanging-togetherness).

$$\mu_\kappa(c, d) = h(\mu_\alpha(c, d), f(c), f(d), c, d)$$

Example

$$\mu_\kappa(c, d) = \mu_\alpha(c, d)\,(w_1 G_1(f(c) + f(d)) + w_2 G_2(f(c) - f(d)))$$

where $G_j(x) = \exp\left(-\frac{1}{2}\frac{(x - m_j)^2}{\sigma_j^2}\right)$

Paths between spels

A **path** p_{cd} in $\mathcal{C}$ from spel $c \in C$ to spel $d \in C$ is any sequence $\langle c_1, c_2, \ldots, c_m\rangle$ of $m \geq 2$ spels in C, where $c_1 = c$ and $c_m = d$.

Let P_{cd} denote the set of all possible paths p_{cd} from c to d. Then the set of all possible paths in $\mathcal{C}$ is

$$P_{\mathcal{C}} = \bigcup_{c,d \in C} P_{cd}$$

Strength of connectedness

The **fuzzy κ-net** $\mathcal{N}_\kappa$ of $\mathcal{C}$ is a fuzzy subset of $P_{\mathcal{C}}$, where the membership (**strength of connectedness**) assigned to any path $p_{cd} \in P_{cd}$ is the smallest spel affinity along p_{cd}

$$\mu_{\mathcal{N}_\kappa}(p_{cd}) = \min_{j=1,\ldots,m-1} \mu_\kappa(c_j, c_{j+1})$$

The **fuzzy κ-connectedness** in $\mathcal{C}$ (K) is a fuzzy relation in C and assigns a value to a pair of spels (c, d) that is the maximum of the strengths of connectedness assigned to all possible paths from c to d (global hanging-togetherness).

$$\mu_K(c, d) = \max_{p_{cd} \in P_{cd}} \mu_{\mathcal{N}_\kappa}(p_{cd})$$

Theorem (Udupa and Samarasekera, 1996):

Fuzzy connectedness is a similitude relation.

Fuzzy κ_θ component

Let $\theta \in [0, 1]$ be a given threshold

Let K_θ be the following binary (equivalence) relation in C

$$\mu_{K_\theta}(c, d) = \begin{cases} 1 & \text{if } \mu_\kappa(c, d) \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

Let $O_\theta(o)$ be the equivalence class of K_θ that contains $o \in C$

Let $\Omega_\theta(o)$ be defined over the fuzzy κ-connectedness K as

$$\Omega_\theta(o) = \{c \in C \,|\, \mu_K(o, c) \geq \theta\}$$

Practical computation of FC relies on the following equivalence

$$O_\theta(o) = \Omega_\theta(o)$$

Fuzzy connected object

The **fuzzy κ_θ object** $\mathcal{O}_\theta(o)$ of $\mathcal{C}$ containing o is

$$\mu_{\mathcal{O}_\theta(o)}(c) = \begin{cases} \eta(c) & \text{if } c \in O_\theta(o) \\ 0 & \text{otherwise} \end{cases}$$

that is

$$\mu_{\mathcal{O}_\theta(o)}(c) = \begin{cases} \eta(c) & \text{if } c \in \Omega_\theta(o) \\ 0 & \text{otherwise} \end{cases}$$

where η assigns an objectness value to each spel perhaps based on $f(c)$ and $\mu_K(o,c)$.

Fuzzy connected objects are robust to the selection of seeds.

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Fuzzy connectedness as a graph search problem

- Spels $\longrightarrow$ graph nodes
- Spel faces $\longrightarrow$ graph edges
- Fuzzy spel-affinity relation $\longrightarrow$ edge costs
- Fuzzy connectedness $\longrightarrow$ all-pairs shortest-path problem
- Fuzzy connected objects $\longrightarrow$ connected components

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Computing fuzzy connectedness

Dynamic programming

Algorithm

Input: $\mathcal{C}$, $o \in C$, κ
Output: A K-connectivity scene $\mathcal{C}_o = (C_o, f_o)$ of $\mathcal{C}$
Auxiliary data: a queue Q of spels

begin
set all elements of $\mathcal{C}_o$ to 0 except o which is set to 1
push all spels $c \in C_o$ such that $\mu_\kappa(o,c) > 0$ to Q
while $Q \neq \emptyset$ **do**
remove a spel c from Q
$f_{\text{val}} \leftarrow \max_{d \in C_o}[\min(f_o(d), \mu_\kappa(c,d))]$
if $f_{\text{val}} > f_o(c)$ **then**
$f_o(c) \leftarrow f_{\text{val}}$
push all spels e such that $\mu_\kappa(c,e) > 0$ $f_{\text{val}} > f_o(e)$ $f_{\text{val}} > f_o(e)$ and $\mu_\kappa(c,e) > f_o(e)$
endif
endwhile
end

Fuzzy connectedness variants

- Multiple seeds per object

- Scale-based fuzzy affinity
- Vectorial fuzzy affinity

- Absolute fuzzy connectedness
- Relative fuzzy connectedness
- Iterative relative fuzzy connectedness

Components of fuzzy affinity

Fuzzy spel adjacency $\mu_\alpha(c,d)$ indicates the degree of spatial adjacency of spels

The **homogeneity-based component** $\mu_\psi(c,d)$ indicates the degree of local hanging-togetherness of spels due to their similarities of intensities

The **object-feature-based component** $\mu_\phi(c,d)$ indicates the degree of local hanging-togetherness of spels with respect to some given object feature

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Defining fuzzy spel affinity

Fuzzy spel affinity

$$\mu_\kappa(c,d) = \mu_\alpha(c,d)g(\mu_\psi(c,d), \mu_\phi(c,d))$$

Expected properties of g

- range within $[0,1]$
- monotonically non-decreasing in both arguments

Examples

$$\mu_\kappa = \frac{1}{2}\mu_\alpha\left(\mu_\psi + \mu_\phi\right)$$

$$\mu_\kappa = \mu_\alpha\sqrt{\mu_\psi\mu_\phi}$$

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Homogeneity-based component

Non-scale-based

Homogeneity-based component

$$\mu_\psi(c,d) = W_\psi(|f(c) - f(d)|)$$

Expected properties of W_ψ

- range within $[0,1]$ and $W_\psi(0) = 1$
- monotonically non-increasing
- should also be related to overall homogeneity

Examples

the right-hand-side of an appropriately scaled box, trapezoid, or Gaussian function

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Object-feature-based component

Non-scale-based

Object-feature-based component

$$\mu_\phi(c,d) = \begin{cases} 1 & \text{if } c = d \\ \frac{\mathcal{W}_o(c,d)}{\mathcal{W}_b(c,d)+\mathcal{W}_o(c,d)} & \text{otherwise} \end{cases}$$

$$\mathcal{W}_o(c,d) = \min[W_o(f(c)), W_o(f(d))]$$

$$\mathcal{W}_b(c,d) = \max[W_b(f(c)), W_b(f(d))]$$

Expected properties of W_o and W_b

- range within $[0,1]$

Examples

an appropriately scaled and shifted box, trapezoid, or Gaussian function

Scale-based affinity

Considers the following aspects

- spatial adjacency
- homogeneity (local and global)
- object feature (expected intensity properties)
- object scale

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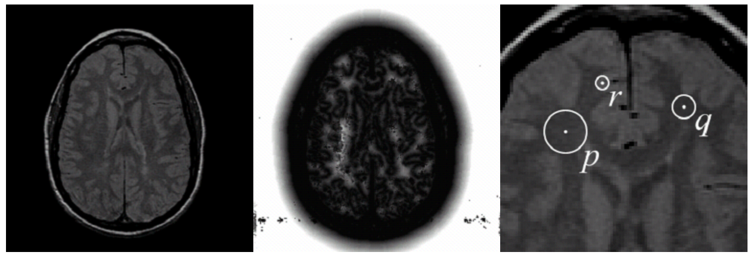
Object scale

Object scale in $\mathcal{C}$ at any spel $c \in C$ is the radius $r(c)$ of the largest hyperball centered at c which lies entirely within the same object region



The scale value can be simply and effectively estimated without explicit object segmentation

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Computing object scale

Algorithm

Input: $\mathcal{C}$, $c \in C$, W_ψ, $\tau \in [0,1]$
Output: $r(c)$

begin
  $k \leftarrow 1$
  while $FO_k(c) \geq \tau$ **do**
    $k \leftarrow k+1$
  endwhile
  $r(c) \leftarrow k$
end

Fraction of the ball boundary homogeneous with the center spel

$$FO_k(c) = \frac{\displaystyle\sum_{d \in B_k(c)} W_{\psi_s}(|f(c) - f(d)|)}{|B_k(c) - B_{k-1}(c)|}$$

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Neighborhood selection for scale-based computations

For properties based on a single spel c use

$$B_r(c) = \{e \in C \mid \|e - c\| \leq r(c)\}$$

For properties based on a pair of spels c and d use

$$B_{cd}(c) = \{e \in C \mid \|e - c\| \leq \min[r(c), r(d)]\}$$

$$B_{cd}(d) = \{e \in C \mid \|e - d\| \leq \min[r(c), r(d)]\}$$

Intensity variations in neighborhoods

Intensity variations (inhomogeneity) surrounding c and d ($\mu_\alpha(c,d) > 0$)

- intra-object variation
  (expected to be random and to have a zero mean)
- inter-object variation
  (expected to have a direction and to be larger than the intra-object variation)

Homogeneity-based component

Scale-based

Directional inhomogeneity over the regions around c and d

$$D(c,d) = \sum_{\substack{e \in B_{cd}(c) \\ e' \in B_{cd}(d) \\ e-c=e'-d}} (1 - W_h(f(e) - f(e')))\,\omega_{cd}(\|e-c\|)\frac{f(e)-f(e')}{|f(e)-f(e')|}$$

Homogeneity-based component

$$\mu_{\psi_s}(c,d) = 1 - \frac{|D(c,d)|}{\sum_{e \in B_{cd}(c)} \omega_{cd}(\|e-c\|)}$$

Object-feature-based component

Scale-based

Filtered version of $f(c)$ taking into account the neighborhood

$$f_a(c) = \frac{\sum_{e \in B_r(c)} f(e)\omega_c(\|e-c\|)}{\sum_{e \in B_r(c)} \omega_c(\|e-c\|)}$$

Object-feature-based component

$$\mu_{\phi_s}(c,d) = \begin{cases} 1 & \text{if } c = d \\ \dfrac{\mathcal{W}_{os}(c,d)}{\mathcal{W}_{bs}(c,d) + \mathcal{W}_{os}(c,d)} & \text{otherwise} \end{cases}$$

$$\mathcal{W}_{os}(c,d) = \min[W_{os}(f_a(c)), W_{os}(f_a(d))]$$

$$\mathcal{W}_{bs}(c,d) = \max[W_{bs}(f_a(c)), W_{bs}(f_a(d))]$$

Computing fuzzy connectedness

Dynamic programming

Algorithm

Input: $\mathcal{C}$, $o \in C$, κ
Output: A K-connectivity scene $\mathcal{C}_o = (C_o, f_o)$ of $\mathcal{C}$
Auxiliary data: a queue Q of spels

begin
  set all elements of $\mathcal{C}_o$ to 0 except o which is set to 1
  push all spels $c \in C_o$ such that $\mu_\kappa(o,c) > 0$ to Q
  while $Q \neq \emptyset$ **do**
    remove a spel c from Q
    $f_{\text{val}} \leftarrow \max_{d \in C_o}[\min(f_o(d), \mu_\kappa(c,d))]$
    if $f_{\text{val}} > f_o(c)$ **then**
      $f_o(c) \leftarrow f_{\text{val}}$
      push all spels e such that $\mu_\kappa(c,e) > 0$ $f_{\text{val}} > f_o(e)$ $f_{\text{val}} > f_o(e)$ and $\mu_\kappa(c,e) > f_o(e$
    endif
  endwhile
end

Computing fuzzy connectedness

Dijkstra's-like

Algorithm

Input: $\mathcal{C}$, $o \in C$, κ
Output: A K-connectivity scene $\mathcal{C}_o = (C_o, f_o)$ of $\mathcal{C}$
Auxiliary data: a priority queue Q of spels

```
begin
  set all elements of C_o to 0 except o which is set to 1
  push o to Q
  while Q ≠ ∅ do
    remove a spel c from Q for which f_o(c) is maximal
    for each spel e such that μ_κ(c, e) > 0 do
      f_val ← min(f_o(c), μ_κ(c, e))
      if f_val > f_o(e) then
        f_o(e) ← f_val
        update e in Q (or push if not yet in)
      endif
    endfor
  endwhile
end
```


Vector-valued scenes

Scene

$$\mathcal{C} = (C, \mathbf{f}) \quad \text{where } C \subset Z^n \text{ and } \mathbf{f} = (f_1, f_2, \ldots, f_l)^{\mathsf{T}}$$

$$\mathbf{f} : C \to [L_1, H_1] \times [L_2, H_2] \times \cdots \times [L_l, H_l]$$

Object scale

$$FO_k(c) = \frac{\displaystyle\sum_{d \in B_k(c)} W_h(\mathbf{f}(c) - \mathbf{f}(d))}{|B_k(c) - B_{k-1}(c)|}$$

$$W_h(\mathbf{x}) = \exp\left(-\frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{\Sigma}_h^{-1}\mathbf{x}\right)$$

Homogeneity-based component

Vectorial scale-based

Total unidirectional inhomogeneity over the scale regions around c and d

$$\mathbf{D}(c,d) = \sum_{\substack{e \in B_{cd}(c) \\ e' \in B_{cd}(d) \\ e-c=e'-d}} (1 - W_h(\mathbf{f}(e) - \mathbf{f}(e')))\omega_{cd}(\|e-c\|)\frac{\mathbf{f}(e) - \mathbf{f}(e')}{|\mathbf{f}(e) - \mathbf{f}(e')|}$$

$$\mu_{\psi_{vs}}(c,d) = 1 - \frac{|\mathbf{D}(c,d)|}{\displaystyle\sum_{e \in B_{cd}(c)} \omega_{cd}(\|e-c\|)}$$

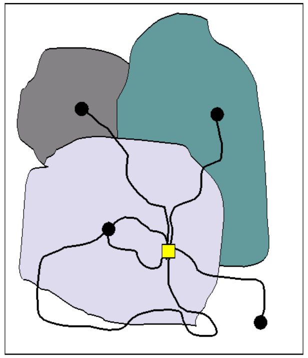
Object-feature-based component

Vectorial scale-based

$$W_o(\mathbf{f}_a(c)) = \exp\left(-\frac{1}{2}(\mathbf{f}_a(c) - \mathbf{M}_o)^{\mathsf{T}}\mathbf{\Sigma}_o^{-1}(\mathbf{f}_a(c) - \mathbf{M}_o)\right)$$

$$\mu_{\phi_{vs}}(c,d) = \begin{cases} 1 & \text{if } c = d \\ \min[W_o(\mathbf{f}_a(c)), W_o(\mathbf{f}_a(d))] & \text{otherwise} \end{cases}$$

Relative fuzzy connectedness



- always at least two objects
- automatic/adaptive thresholds on the object boundaries
- objects (object seeds) "compete" for spels and the one with stronger connectedness wins

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Relative fuzzy connectedness

Let $O_1, O_2, \dots, O_m$, a given set of objects ($m \geq 2$), $S = \{o_1, o_2 \dots, o_m\}$ a set of corresponding seeds, and let $b(o_j) = S \setminus \{o_j\}$ denote the 'background' seeds w.r.t. seed o_j.

1. define affinity for each object $\Rightarrow \kappa_1, \kappa_2, \dots, \kappa_m$
2. combine them into a single affinity $\Rightarrow \kappa = \bigcup_j \kappa_j$
3. compute fuzzy connectedness using $\kappa \Rightarrow K$
4. determine the fuzzy connected objects $\Rightarrow$

$$O_{ob}(o) = \{c \in C \,|\, \forall o' \in b(o) \quad \mu_K(o,c) > \mu_K(o',c)\}$$

$$\mu_{\mathcal{O}_{ob}}(c) = \begin{cases} \eta(c) & \text{if } c \in O_{ob}(o) \\ 0 & \text{otherwise} \end{cases}$$

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kNN vs. VSRFC

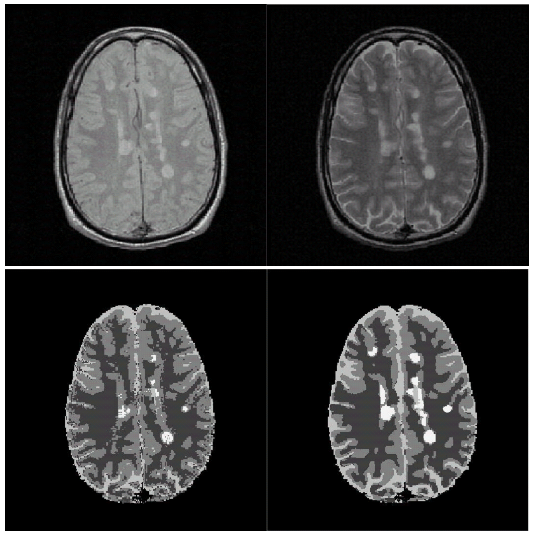

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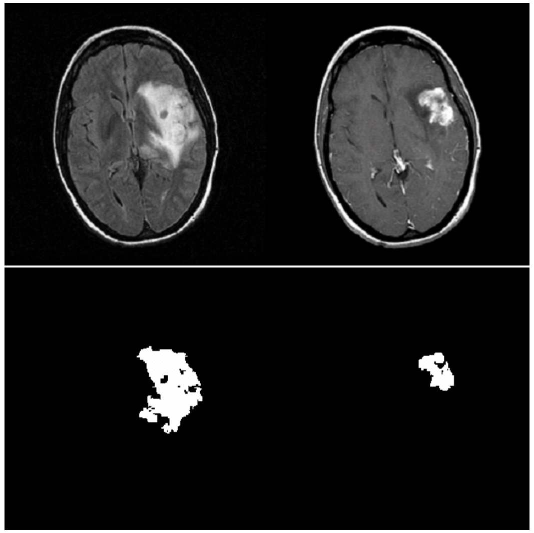
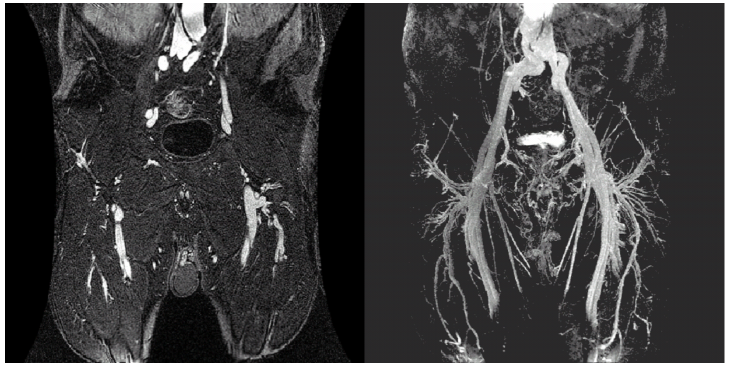
Image segmentation using FC

- MR
  - brain tissue, tumor, MS lesion segmentation
- MRA
  - vessel segmentation and artery-vein separation
- CT bone segmentation
  - kinematics studies
  - measuring bone density
  - stress-and-strain modeling
- CT soft tissue segmentation
  - cancer, cyst, polyp detection and quantification
  - stenosis and aneurism detection and quantification
- Digitized mammography
  - detecting microcalcifications
- Craniofacial 3D imaging
  - visualization and surgical planning

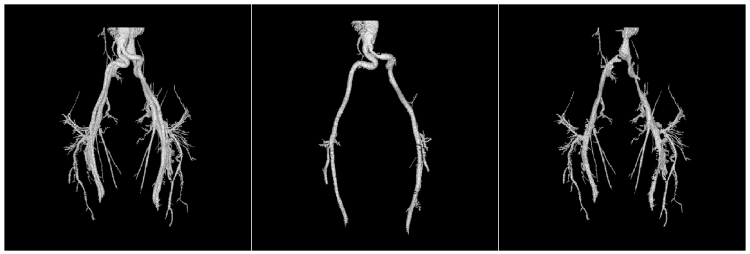
Brain tumor quantification



MRA slice and MIP rendering



MRA vessel segmentation and artery/vein separation



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