Fuzzy Sets and Fuzzy Techniques

Lecture 10 – Fuzzy Logic and Approximate Reasoning

Nataša Sladoje

Centre for Image Analysis
Uppsala University

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Fuzzy Sets and Fuzzy Techniques
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Introduction
Fuzzy Logic
Fuzzy Implications
Binary Fuzzy Relations
Approximate Reasoning

Outline

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Introduction

- Classical logic - a very brief overview — Chapter 8.1
- Multivalued logic — Chapter 8.2

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Classical logic: A brief overview

Propositional logic

- **Logic** is the study of methods and principles of **reasoning** in all its possible forms.
- **Propositions** - statements that are required to be **true** or **false**.
- The **truth value** of a proposition is the opposite of the truth value of its **negation**.
- Instead of propositions, we use **logic variables**. Logic variable may asses one of the two truth values, if it is substituted by a particular proposition.
- **Propositional logic** studies the rules by which new logic variables can be produced from some given logic variables. The internal structure of the propositions "behind" the variables does not matter!

Classical logic: A brief overview

Logic functions

Logic function assigns a truth value to a combination of truth values of its variables:

$$f : \{true, false\}^n \to \{true, false\}$$

2^n choices of n arguments $\to$ 2^{2^n} logic functions of n variables.

Classical logic: A brief overview

Logic functions of two variables

v_2 v_1	1 1	1 0	0 1	0 0	Function name	Adopted symbol
ω_1	0	0	0	0	Zero function	$\mathbf{0}$
ω_2	0	0	0	1	NOR function	$v_1 \downarrow v_2$
ω_3	0	0	1	0	Inhibition	$v_1 > v_2$
ω_4	0	0	1	1	Negation	$\bar{v}_2$
ω_5	0	1	0	0	Inhibition	$v_1 < v_2$
ω_6	0	1	0	1	Negation	$\bar{v}_1$
ω_7	0	1	1	0	Exclusive OR	$v_1 \oplus v_2$
ω_8	0	1	1	1	NAND function	$v_1 \mid v_2$
ω_9	1	0	0	0	Conjunction	$v_1 \wedge v_2$
ω_{10}	1	0	0	1	Equivalence	$v_1 \Leftrightarrow v_2$
ω_{11}	1	0	1	0	Assertion	v_1
ω_{12}	1	0	1	1	Implication	$v_1 \Leftarrow v_2$
ω_{13}	1	1	0	0	Assertion	v_2
ω_{14}	1	1	0	1	Implication	$v_1 \Rightarrow v_2$
ω_{15}	1	1	1	0	Disjunction	$v_1 \vee v_2$
ω_{16}	1	1	1	1	One function	$\mathbf{1}$

Classical logic: A brief overview

Logic primitives

- Observe, e.g.:

$$\omega_{14}(v_1, v_2) = \omega_{15}(\omega_6(v_1, v_2), v_2)$$
$$\omega_{10}(v_1, v_2) = \omega_9(\omega_{14}(v_1, v_2), \omega_{12}(v_1, v_2))$$

- **A task**: Express all the logic functions of n variables by using only a small number of simple logic functions, preferably of one or two variables.
- Such a set is **a complete set of logic primitives**.
- Examples:

$\{negation, conjunction, disjunction\} = \{\omega_6, \omega_9, \omega_{15}\}$,
$\{negation, implication\} = \{\omega_6, \omega_{14}\}$.

Classical logic: A brief overview

Logic formulae

Definition

1. If v is a logic variable, then v and $\bar{v}$ are logic formulae;
2. If v_1 and v_2 are logic formulae, then $v_1 \wedge v_2$ and $v_1 \vee v_2$ are also logic formulae;
3. Logic formulae are only those defined (obtained) by the two previous rules.

Classical logic: A brief overview

Logic formulae

Each logic formula generates a unique logic function. Different logic formulae may generate the same logic function. Such are called **equivalent**.

Examples:

$$(v_1 \Rightarrow v_2) \quad \Leftrightarrow \quad (\bar{v}_1 \vee v_2)$$
$$(v_1 \Leftrightarrow v_2) \quad \Leftrightarrow \quad ((v_1 \Rightarrow v_2) \wedge (v_1 \Leftarrow v_2))$$

Tautology is (any) logic formula that corresponds to a logic function **one**.
Contradiction is (any) logic formula that corresponds to a logic function **zero**.

Classical logic: A brief overview

Inference rules

Inference rules are tautologies used for making deductive inferences.

Examples:

- $(a \wedge (a \Rightarrow b)) \Rightarrow b$ **modus ponens**
- $(\bar{b} \wedge (a \Rightarrow b)) \Rightarrow \bar{a}$ **modus tollens**
- $(a \Rightarrow b) \wedge (b \Rightarrow c)) \Rightarrow (a \Rightarrow c)$ **hypothetical syllogism**

Classical logic: A brief overview

Predicate logic

- There are situations when the internal structure of propositions cannot be ignored in deductive reasoning.
- Propositions are, in general, of the form

  $$x \text{ is } P$$

  where x is a symbol of a **subject** and P is a **predicate** that characterizes a property.
- x is any element of universal set X, while P is a function on X, which for each value of x forms a proposition.
- $P(x)$ is called **predicate**; it becomes true or false for any particular value of x.

Classical logic: A brief overview

Predicate logic-extensions

- n-ary predicates $P(x_1, x_2, \ldots, x_n)$
- Quantification of applicability of a predicate with respect to the domain of its variables
  Existential quantification: $(\exists x)P(x)$
  Universal quantification: $(\forall x)P(x)$
- It holds:

$$(\exists x)P(x) = \bigvee_{x \in X} P(x) \qquad (\forall x)P(x) = \bigwedge_{x \in X} P(x)$$

Multivalued Logics

Three-valued logic

- Third truth value is allowed:
  truth: 1, false: 0, intermediate: $\frac{1}{2}$.
- While it is accepted to have $\bar{p} = 1 - p$, the definitions of other primitives differ in different three-valued logics.
- For the best known three-valued logics (Łukasiewicz, Bochvar, Kleene, Heyting, Reichenbach), primitives coincide with two valued counterparts for the variables having values 0 or 1 (see Table 8.4, p.218).
- None of the mentioned logics satisfies law of excluded middle, or law of contradiction.
- **quasi-tautology** is a logic formula that never assumes truth value 0;
- **quasi-contradiction** is a logic formula that never assumes truth value 1.

Multivalued Logics

n-valued logic

The set of truth values is

$$T_n = \{0 = \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1\}.$$

Truth values are interpreted as **degrees of truth**.
Primitives in *n*-valued logics of Łukasiewicz, denoted by L_n, are:

$$\begin{aligned} \bar{p} &= 1 - p \\ p \wedge q &= \min[p, q] \\ p \vee q &= \max[p, q] \\ p \Rightarrow q &= \min[1, 1 + q - p] \\ p \Leftrightarrow q &= 1 - |p - q| \end{aligned}$$

Multivalued Logics

Łukasiewicz *n*-valued logic

- L_2 is a classical two valued logic.
- L_∞ takes truth values to be all rational numbers in $[0, 1]$.
- By L_1, the logic with truth values being all real numbers in $[0, 1]$ is denoted.
  It is called **the standard Łukasiewicz logic**.
  It is **isomorphic** with a fuzzy set theory based on standard operations.

There exists no finite complete set of logic primitives for any infinite-valued logic. Using a finite set of primitives, only a subset of all logic functions can be defined.

Fuzzy logic

- Fuzzy propositions — Chapter 8.3
- Linguistic hedges — Chapter 8.5
- Fuzzy quantifiers — Chapter 8.4

Fuzzy propositions

The range of truth values of fuzzy propositions is not only $\{0, 1\}$, but $[0, 1]$.
The truth of a fuzzy proposition is a **matter of degree**.

Classification of fuzzy propositions:

- **Unconditional** and **unqualified** propositions
  "The temperature is high."
- **Unconditional** and **qualified** propositions
  "The temperature is high is very true."
- **Conditional** and **unqualified** propositions
  "If the temperature is high, then it is hot."
- **Conditional** and **qualified** propositions
  "If the temperature is high, then it is hot is true."

Linguistic hedges

- **Linguistic hedges** are linguistic terms by which other linguistic terms are modified.
  "Tina is young is true."
  "Tina is **very** young is true."
  "Tina is young is **very** true."
  "Tina is **very** young is **very** true."
- Fuzzy predicates and fuzzy truth values can be modified.
  Crisp predicates cannot be modified.
- Examples of hedges: **very, fairly, extremely**.

Linguistic hedges (modifiers)

For a given predicate F on X and a given linguistic hedge H, a new (modified) fuzzy predicate HF is defined as:

$$HF(x) = h(F(x)), \qquad \text{for all } x \in X.$$

A **modifier** h is a unary operation $h : [0, 1] \to [0, 1]$ such that:

- $h(0) = 0$ and $h(1) = 1$;
- h is a continuous function;
- If $h(a) < a$ for all $a \in [0, 1]$, (i.e., if h is **strong**), then $h^{-1}(a) > a$ for all $a \in [0, 1]$, (i.e., then h^{-1} is **weak**).
- A composition of modifiers is also a modifier.

Strong modifier **reduces** the truth value of a proposition.
Weak modifier **increases** the truth value of a proposition (by weakening the proposition).
An identity modifier is a function $h(a) = a$.

Modifiers

One commonly used class of modifiers is

$$h_\alpha(a) = a^\alpha, \qquad \text{for } \alpha \in R^+ \text{ and } a \in [0, 1].$$

For $\alpha < 1$, h_α is a weak modifier.
Example: H : *fairly* $\leftrightarrow$ $h(a) = \sqrt{a}$.

For $\alpha > 1$, h_α is a strong modifier.
Example: H : *very* $\leftrightarrow$ $h(a) = a^2$.

h_1 is the identity modifier.

Modifiers

Example: Tina is 26.
p_1: Tina is young. $YOUNG(26) = 0.8$
p_2: Tina is very young. $VERY\_YOUNG(26) = 0.8^2 = 0.64$
p_3: Tina is fairly young. $FAIRLY\_YOUNG(26) = \sqrt{0.8} = 0.89$

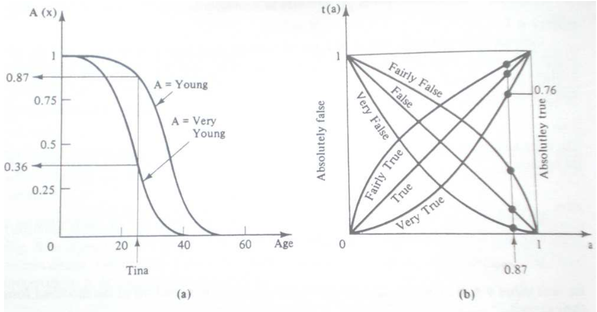


Figure 8.2 Truth values of a fuzzy proposition.

Fuzzy quantifiers

- **Absolute** quantifiers:
  "about 10"; "much more than 100", ...
- **Relative** quantifiers:
  "almost all"; "about half", ...

Examples:
p: "There are **about 3** high-fluent students in the group."
q: "**Almost all** students in the group are high-fluent."

Fuzzy quantifiers

To determine the truth value of a quantified proposition, we need to know

1. "how many" students in the group are high-fluent
   i.e., cardinality of a fuzzy set *High-fluent*
2. "how much" is that value *about 3*
   i.e., membership of the obtained value to the fuzzy set *About 3*

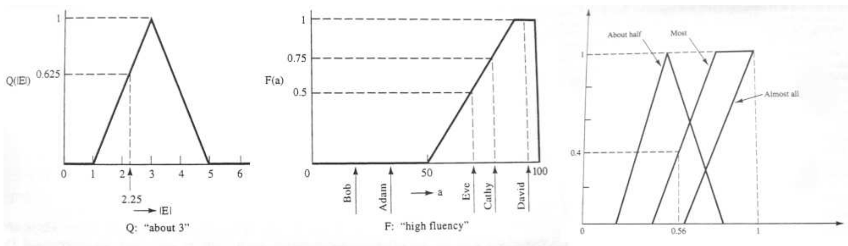
or

1. "how many" students in the group are high-fluent, relatively to the size of the group
   i.e., cardinality of a fuzzy set *High-fluent* divided by the size of the group
2. "how much" is that value *almost all*
   i.e., membership of the obtained value to the fuzzy set *Almost all*.

Fuzzy quantifiers

An example

Group = { Adam, Bob, Cathy, David, Eve }.
Fluency is represented by the value from the interval [0, 100].
Fuzzy set F represents "High fluency" on [0,100].
Fuzzy set Q represents fuzzy quantifier "about 3".



$E = 0/Adam + 0/Bob + 0.75/Cathy + 1/David + 0.5/Eve$
is a fuzzy set "High fluency´´ on the domain *Group*.

$|E| = 2.25$ $T(p) = Q(|E|) = Q(2.25) = 0.625.$

$\frac{|E|}{|Group|} = 0.45$ $T(q) = Q_1(0.45) = 0.$

Fuzzy propositions

Unconditional and unqualified propositions

The canonical form

$$p: \ \nu \ \text{is} \ F$$

ν is a variable on some universal set V
F is a fuzzy set on V that represents a fuzzy predicate
(e.g., low, tall, young, expensive...)

The **degree of truth** of p is

$$T(p) = F(v), \qquad \text{for } v \in \nu.$$

T is a fuzzy set on V. Its membership function is derived form the membership function of a fuzzy predicate F.

The role of a function T is to connect fuzzy sets and fuzzy propositions.

In case of unconditional and unqualified propositions, the identity function is used.

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Fuzzy propositions

Unconditional and unqualified propositions

An illustration:

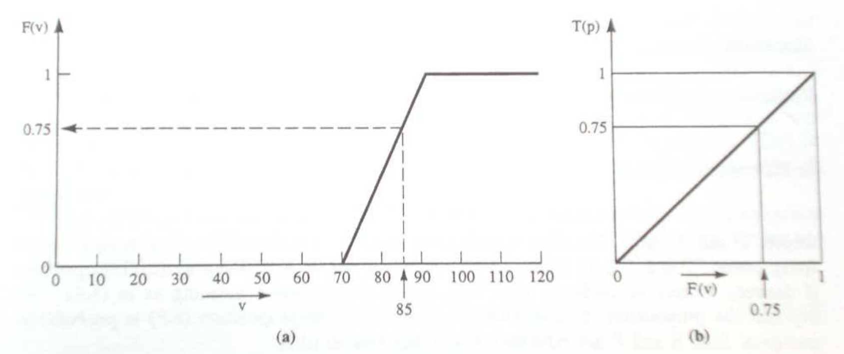

Figure 8.1 Components of the fuzzy proposition p: Temperature (V) is high (F).

Note: The proposition can be expressed as "ν is F is true."

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Fuzzy propositions

Unconditional and qualified propositions

The canonical form

$$p: \ \nu \ \text{is} \ F \ \text{is} \ S \qquad \text{(truth qualified proposition)}$$

where ν is a variable on some universal set V,
F is a fuzzy set on V that represents a fuzzy predicate,
and S is a **fuzzy truth qualifier**.

To calculate the degree of truth $T(p)$ of the proposition p, we use:

$$T(p) = S(F(v))$$

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Fuzzy propositions

Unconditional and qualified propositions

An illustration:

p: " Tina is young is very true".
Tina is 26.

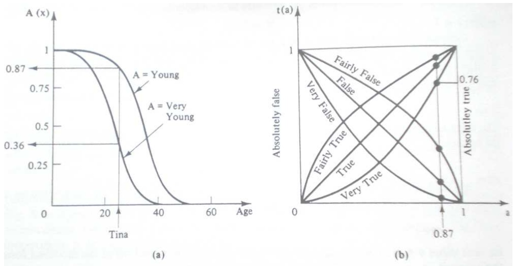

Figure 8.2 Truth values of a fuzzy proposition.

$$Young(26) = 0.87, \quad \text{and} \quad VeryTrue(0.87) = 0.76$$

$$T(p) = 0.76.$$

Fuzzy propositions

Conditional and unqualified propositions

The canonical form

$$p: \text{ If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B,$$

where $\mathcal{X}, \mathcal{Y}$ are variables on X, Y respectively, and A, B are fuzzy sets on X, Y respectively.

Alternative form:

$$\langle \mathcal{X}, \mathcal{Y} \rangle \quad \text{is} \quad R$$

where $R(x, y) = \mathcal{J}(A(x), B(x))$ is a fuzzy set on $X \times Y$ representing a suitable fuzzy implication.

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Conditional and qualified propositions

The canonical form

$$p: \text{ If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B \text{ is } S$$

where $\mathcal{X}, \mathcal{Y}$ are variables on X, Y respectively, A, B are fuzzy sets on X, Y respectively, and S is a truth qualifier.

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Fuzzy implications

- Fuzzy implications — Chapter 11.2
- Selection of fuzzy implications — Chapter 11.3

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Fuzzy implications

Definition(s)

A **fuzzy implication** $\mathcal{J}$ of two fuzzy propositions p and q is a function of the form

$$\mathcal{J}: [0,1] \times [0,1] \to [0,1],$$

which for any truth values $a = T(p)$ and $b = T(q)$ defines the truth value $\mathcal{J}(a, b)$ of the conditional proposition "if p, then q".

Fuzzy implications as extensions of the classical logic implication:

	Crisp implication $a \Rightarrow b$	Fuzzy implication $\mathcal{J}(a, b)$
(S)	$\bar{a} \vee b$	$u(c(a), b)$
(R)	$\max\{x \in \{0,1\} \mid a \wedge x \leq b\}$	$\sup\{x \in [0,1] \mid i(a, x) \leq b\}$
(QL)	$\bar{a} \vee (a \wedge b)$	$u(c(a), i(a, b))$
(QL)	$(\bar{a} \wedge \bar{b}) \vee b$	$u(i(c(a), c(b)), b)$

Fuzzy implications

Axiomatic requirements

Ax1. $a \le b$ implies $\mathcal{J}(a,x) \ge \mathcal{J}(b,x)$ **monotonicity in first argument**

Ax2. $a \le b$ implies $\mathcal{J}(x,a) \le \mathcal{J}(x,b)$ **monotonicity in sec. arg.**

Ax3. $\mathcal{J}(0,a) = 1$ **dominance of falsity**

Ax4. $\mathcal{J}(1,b) = b$ **neutrality of truth**

Ax5. $\mathcal{J}(a,a) = 1$ **identity**

Ax6. $\mathcal{J}(a,\mathcal{J}(b,x)) = \mathcal{J}(b,\mathcal{J}(a,x))$ **exchange property**

Ax7. $\mathcal{J}(a,b) = 1$ iff $a \le b$ **boundary condition**

Ax8. $\mathcal{J}(a,b) = \mathcal{J}(c(b),c(a))$ **contraposition**

Ax9. $\mathcal{J}$ is a continuous function **continuity**

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Examples

TABLE 11.1 LIST OF FUZZY IMPLICATIONS

Name	Symbol	Class	Function $\mathcal{J}(a,b)$	Axioms	Complement $c(a)$	Year
Early Zadeh	$\mathcal{J}_m$	QL	$\max[1-a, \min(a,b)]$	1, 2, 3, 4, 9	$1-a$	1973
Gaines-Rescher	$\mathcal{J}_s$		$\begin{cases} 1 & a \le b \\ 0 & a > b \end{cases}$	1, 2, 3, 4, 5, 6, 7, 8		1969
Gödel	$\mathcal{J}_g$	R	$\begin{cases} 1 & a \le b \\ b & a > b \end{cases}$	1, 2, 3, 4, 5, 6, 7		1976
Goguen	$\mathcal{J}_\Delta$	R	$\begin{cases} 1 & a \le b \\ b/a & a > b \end{cases}$	1, 2, 3, 4, 5, 6, 7, 9		1969
Kleene-Dienes	$\mathcal{J}_b$	S, QL	$\max(1-a, b)$	1, 2, 3, 4, 6, 8, 9	$1-a$	1938, 1949
Lukasiewicz	$\mathcal{J}_a$	R, S	$\min(1, 1-a+b)$	1, 2, 3, 4, 5, 6, 7, 8, 9	$1-a$	1920
Pseudo-Lukasiewicz 1	$\mathcal{J}_\lambda$ $(\lambda > -1)$	R, S	$\min\left[1, \frac{1-a+(1+\lambda)b}{1+\lambda a}\right]$	1, 2, 3, 4, 5, 6, 7, 8, 9	$\frac{1-a}{1+\lambda a}$	1987
Pseudo-Lukasiewicz 2	$\mathcal{J}_w$ $(w > 0)$	R, S	$\min\left[1, (1-a^w+b^w)^{\frac{1}{w}}\right]$	1, 2, 3, 4, 5, 6, 7, 8, 9	$(1-a^w)^{\frac{1}{w}}$	1987
Reichenbach	$\mathcal{J}_r$	S	$1-a+ab$	1, 2, 3, 4, 6, 8, 9	$1-a$	1935
Willmott	$\mathcal{J}_{wi}$		$\min[\max(1-a,b), \max(a,1-a), \max(b,1-b)]$	4, 6, 8, 9	$1-a$	1980
Wu	$\mathcal{J}_{wu}$		$\begin{cases} 1 & a \le b \\ \min(1-a,b) & a > b \end{cases}$	1, 2, 3, 5, 7, 8	$1-a$	1986
Yager	$\mathcal{J}_y$		$\begin{cases} 1 & a = b = 0 \\ b^a & \text{others} \end{cases}$	1, 2, 3, 4, 6		1980
Klir and Yuan 1	$\mathcal{J}_p$	QL	$1-a+a^2b$	2, 3, 4, 9	$1-a$	1994
Klir and Yuan 2	$\mathcal{J}_q$	QL	$\begin{cases} b & a = 1 \\ 1-a & a \ne 1, b \ne 1 \\ 1 & a \ne 1, b = 1 \end{cases}$	2, 4	$1-a$	1994

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How to select fuzzy implication

Criteria related to fuzzy inference rules
modus ponens, modus tollens, hypothetical syllogism.

Idea: If reduced to crisp sets, these rules should coincide with corresponding classical inference rules.

More formally: for fuzzy sets $A(x)$, $B(y)$ representing truth values by membership grades in [0,1]

$$\begin{aligned} B(y) &= \sup_{x \in X} i(A(x), \mathcal{J}(A(x), B(y))) \quad \text{modus ponens} \\ c(A(x)) &= \sup_{y \in Y} i(c(B(y)), \mathcal{J}(A(x), B(y))) \quad \text{modus tollens} \\ \mathcal{J}(A(x), C(z)) &= \sup_{y \in Y} i(\mathcal{J}(A(x), B(y)), \mathcal{J}(B(y), C(z))) \quad \text{hypothet. syllog.} \end{aligned}$$

should hold.

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How to select fuzzy implication

Look at Table 11.2 , Table 11.3, and Table 11.4 (pp. 315-317).

One good choice:

$$\mathcal{J}_s(a,b) = \begin{cases} 1 & a \le b \\ 0 & a > b \end{cases}$$

One frequently used implication: **Łukasiewicz**

$$\mathcal{J}_a(a,b) = \min[1, 1-a+b]$$

Binary fuzzy relations

- Binary fuzzy relations – definition   Chapter 5.3

Binary fuzzy relations

A super-brief introduction

- A **crisp** binary relation R on sets X, Y is any (crisp) subset of $X \times Y$.
- xRy
  ($x \in X$ is in relation R with $y \in Y$) iff $(x, y) \in R$
- A **fuzzy** binary relation R on sets X, Y is any **fuzzy** subset of $X \times Y$.
- Elements $x \in X$ and $y \in Y$ are in relation R up to some extent.

Binary fuzzy relations

A super-brief introduction

Terminology: for a given fuzzy relation $R(X, Y)$

- **domain** of R is $domR(x) = \max_{y \in Y} R(x, y)$, for each $x \in X$.
- **range** of R is $ranR(y) = \max_{x \in X} R(x, y)$, for each $y \in Y$.
- **height** of R is $h(R) = \max_{y \in Y} \max_{x \in X} R(x, y)$.

The **standard composition** of two fuzzy relations, $P(X, Y)$ and $Q(Y, Z)$, is a binary relation $R(X, Z)$ defined by

$$R(x, z) = [P \circ Q](x, z) = \max_{y \in Y} \min[P(x, y), Q(y, z)]$$

for all $x \in X$ and all $z \in Z$.

This composition is based on standard t-norm, and standard t-conorm. It is also referred to as **max-min composition**.

Binary fuzzy relations

A super-brief introduction

To represent (fuzzy) binary relations, **membership matrices** are convenient.

$$R = [r_{xy}], \quad \text{where } r_{xy} = R(x, y).$$

An example:

Two fuzzy binary relations, $P(X, Y)$ and $Q(Y, Z)$ are given:

$$P = \begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0.0 & 0.7 & 1.0 \\ 0.4 & 0.6 & 0.5 \end{bmatrix} \qquad Q = \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0.0 & 0.9 \\ 1.0 & 0.0 & 0.5 & 0.5 \end{bmatrix}.$$

We read that, e.g.,

$dom\ P(x_2) = \max[0.0, 0.7, 1.0] = 1.0,$

$ran\ Q(y_3) = \max[0.7, 0.0, 0.5] = 0.7.$

Binary fuzzy relations

A super-brief introduction

We can also determine
$R = P \circ Q = [r_{ij}] = [p_{ik}] \circ [q_{kj}] = [\max_k \min(p_{ik}, q_{kj})]$

$$R = P \circ Q = \begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0.0 & 0.7 & 1.0 \\ 0.4 & 0.6 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0.0 & 0.9 \\ 1.0 & 0.0 & 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.3 & 0.5 & 0.5 \\ 1.0 & 0.2 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.6 \end{bmatrix}.$$

For example

$$r_{23} = \max[\min(0.0, 0.7), \min(0.7, 0.0), \min(1.0, 0.5)] = \max[0.0, 0.0, 0.5] = 0.5.$$

Approximate reasoning

- Inference rules from conditional fuzzy propositions
  Chapter 8.6
- Multiconditional approximate reasoning
  Chapter 11.4

Inference rules

Fuzzy inference rules are basis for **approximate reasoning**.

As an example, three classical inference rules
(Modus ponens, Modus Tollens, Hypothetical syllogism)
are generalized by using **compositional rule of inference**

For a given fuzzy relation R on $X \times Y$, and a given fuzzy set A' on X, a fuzzy set B' on Y can be derived for all $y \in Y$, so that

$$B'(y) = \sup_{x \in X} \min[A'(x), R(x, y)].$$

In matrix form, compositional rule of inference is

$$\mathbf{B}' = \mathbf{A}' \circ \mathbf{R}$$

Inference rules

Fuzzy propositions as relations

The fuzzy relation **R** is, e.g., given by (one or more) conditional fuzzy propositions.

For a given fuzzy proposition

$$p: \text{ If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B$$

a corresponding fuzzy relation is

$$R(x, y) = \mathcal{J}[A(x), B(y)], \qquad \text{for all } x \in X, y \in Y$$

where $\mathcal{J}$ stands for a fuzzy implication.

Inference rules

Generalized modus ponens

Rule:	If $\mathcal{X}$ is A, then $\mathcal{Y}$ is B
Fact:	$\mathcal{X}$ is A'
Conclusion:	$\mathcal{Y}$ is B'

In this case,

$$R(x,y) = \mathcal{J}[A(x), B(y)]$$

and

$$B'(y) = \sup_{x\in X} \min[A'(x), R(x,y)].$$

Inference rules

Generalized modus ponens – an example

Example:
Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be the sets of values of variables $\mathcal{X}, \mathcal{Y}$.
Let $A = 0.5/x_1 + 1/x_2 + 0.6/x_3$ and $B = 1/y_1 + 0.4/y_2$.
Let $A' = 0.6/x_1 + 0.9/x_2 + 0.7/x_3$.
Let $R(x,y) = \mathcal{J}[A(x), B(y)] = \min[1, 1 - A(x) + B(y)]$.
By using Generalized modus ponens, derive the conclusion $\mathcal{Y}$ is B'.

We compute:

$$\begin{aligned}
R &= 1/x_1,y_1 + 0.9/x_1,y_2 + 1/x_2,y_1 + 0.4/x_2,y_2 + 1/x_3,y_1 + 0.8/x_3,y_2 \\
B'(y_1) &= \sup_{x\in X} \min[A'(x), R(x,y_1)] \\
&= \max[\min(0.6,1), \min(0.9,1), \min(0.7,1)] \\
&= \max[0.6, 0.9, 0.7] = 0.9 \\
B'(y_2) &= \sup_{x\in X} \min[A'(x), R(x,y_2)] \\
&= \max[\min(0.6,0.9), \min(0.9,0.4), \min(0.7,0.8)] \\
&= \max[0.6, 0.4, 0.7] = 0.7
\end{aligned}$$

We conclude that $B' = 0.9/y_1 + 0.7/y_2$.

Inference rules

Generalized modus tollens

Rule:	If $\mathcal{X}$ is A, then $\mathcal{Y}$ is B
Fact:	$\mathcal{Y}$ is B'
Conclusion:	$\mathcal{X}$ is A'

In this case,

$$R(x,y) = \mathcal{J}[A(x), B(y)]$$

and

$$A'(x) = \sup_{y\in Y} \min[B'(y), R(x,y)].$$

Inference rules

Generalized modus tollens – an example

Example:
Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be the sets of values of variables $\mathcal{X}, \mathcal{Y}$.
Let $A = 0.5/x_1 + 1/x_2 + 0.6/x_3$ and $B = 1/y_1 + 0.4/y_2$.
Let $B' = 0.9/y_1 + 0.7/y_2$.
Let $R(x,y) = \mathcal{J}[A(x), B(y)] = \min[1, 1 - A(x) + B(y)]$.
By using Generalized modus tollens, derive the conclusion $\mathcal{X} is A'$.

We compute:

$$\begin{aligned}
R &= 1/x_1,y_1 + 0.9/x_1,y_2 + 1/x_2,y_1 + 0.4/x_2,y_2 + 1/x_3,y_1 + 0.8/x_3,y_2. \\
A'(x_1) &= \sup_{y\in Y} \min[B'(y), R(x_1,y)] \\
&= \max[\min(0.9,1), \min(0.7,0.9)] = \max[0.9, 0.7] = 0.9 \\
A'(x_2) &= \sup_{y\in Y} \min[B'(y), R(x_2,y)] \\
&= \max[\min(0.9,1), \min(0.7,0.4)] = \max[0.9, 0.4] = 0.9 \\
A'(x_3) &= \sup_{y\in Y} \min[B'(y), R(x_3,y)] \\
&= \max[\min(0.9,1), \min(0.7,0.8)] = \max[0.9, 0.7] = 0.9
\end{aligned}$$

We conclude that $A' = 0.9/x_1 + 0.9/x_2 + 0.9/x_3$.

Inference rules

Generalized hypothetical syllogism

For variables $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ taking values from sets X, Y, Z respectively, and A, B, C being fuzzy sets on X, Y, Z, respectively:

Rule 1:	If $\mathcal{X}$ is A, then $\mathcal{Y}$ is B
Rule 2:	If $\mathcal{Y}$ is B, then $\mathcal{Z}$ is C
Conclusion:	If $\mathcal{X}$ is A, then $\mathcal{Z}$ is C

In this case, three relations are defined:

$$\begin{aligned} R_1(x,y) &= \mathcal{J}[A(x), B(y)] \\ R_2(y,z) &= \mathcal{J}[B(y), C(z)] \\ R_3(x,z) &= \mathcal{J}[A(x), C(z)]. \end{aligned}$$

The generalized hypothetical syllogism holds if

$$R_3(x,z) = \sup_{y \in Y} \min[R_1(x,y), R_2(x,y)]$$

or, in matrix notation, if

$$\mathbf{R_3} = \mathbf{R_1} \circ \mathbf{R_2}.$$

Inference rules

Generalized hypothetical syllogism

Example:
Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2\}$ be the sets of values of variables $\mathcal{X}, \mathcal{Y}, \mathcal{X}$.

Let $A = 0.5/x_1 + 1/x_2 + 0.6/x_3$,
$B = 1/y_1 + 0.4/y_2$
$C = 0.2/z_1 + 1/z_2$.

Let

$$R(x,y) = \mathcal{J}[A(x), B(y)] = \begin{cases} 1 & a \le b \\ b & a > b \end{cases}.$$

Check if generalized hypothetical syllogism holds.

We write

$$R_1 = \begin{bmatrix} 1 & 0.4 \\ 1 & 0.4 \\ 1 & 0.4 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.2 & 1 \\ 0.2 & 1 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0.2 & 1 \\ 0.2 & 1 \\ 0.2 & 1 \end{bmatrix}$$

and we check that $\mathbf{R_1} \circ \mathbf{R_2} = \mathbf{R_3}$.

Multiconditional approximate reasoning

General schema is of the form:

Rule 1:	If $\mathcal{X}$ is A_1, then $\mathcal{Y}$ is B_1
Rule 2:	If $\mathcal{X}$ is A_2, then $\mathcal{Y}$ is B_2
	...
Rule n:	If $\mathcal{X}$ is A_n, then $\mathcal{Y}$ is B_n
Fact:	$\mathcal{X}$ is A'
Conclusion:	$\mathcal{Y}$ is B'

A', A_j are fuzzy sets on X,
B', B_j are fuzzy sets on Y, for all j.

Multiconditional approximate reasoning

Method of interpolation

Most common way to determine B' is by using **method of interpolation**.

Step 1. Calculate the degree of consistency between the given fact and the antecedent of each rule.
Use height of intersection of the associated sets:

$$r_j(A') = h(A' \wedge A_j) = \sup_{x \in X} \min[A'(x), A_j(x)].$$

Step 2. Truncate each B_j by the value $r_j(A')$ and determine B' as the union of truncated sets:

$$B'(y) = \sup_{j \in \mathbb{N}_n} \min[r_j(A'), B_j(y)], \quad \text{for all } y \in Y.$$

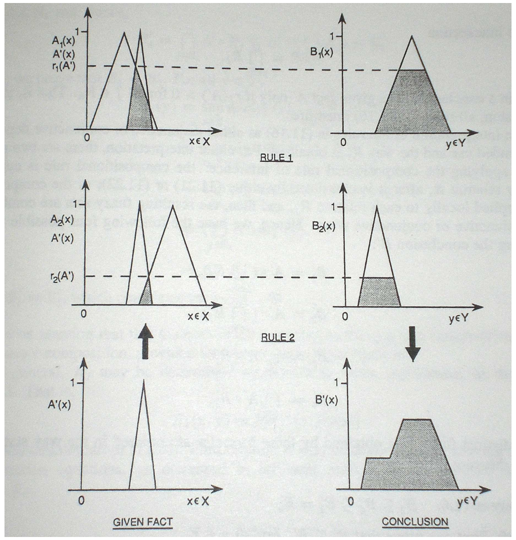
Note that interpolation method is a special case of the composition rule of inference, with

$$R(x,y) = \sup_{j \in \mathbb{N}_n} \min[A_j(x), B_j(y)]$$

where then $B'(y) = \sup_{x \in X} \min[A'(x), R(x,y)] = (A' \circ R)(y)$.

Multiconditional approximate reasoning

Method of interpolation-Example



An application

Region growing based on fuzzy rule based system

A. Steudel and M.Glesner: “Fuzzy segmented image coding using orthonormal bases and derivative chain coding”, *Pattern Recognition*, 32, 1999.

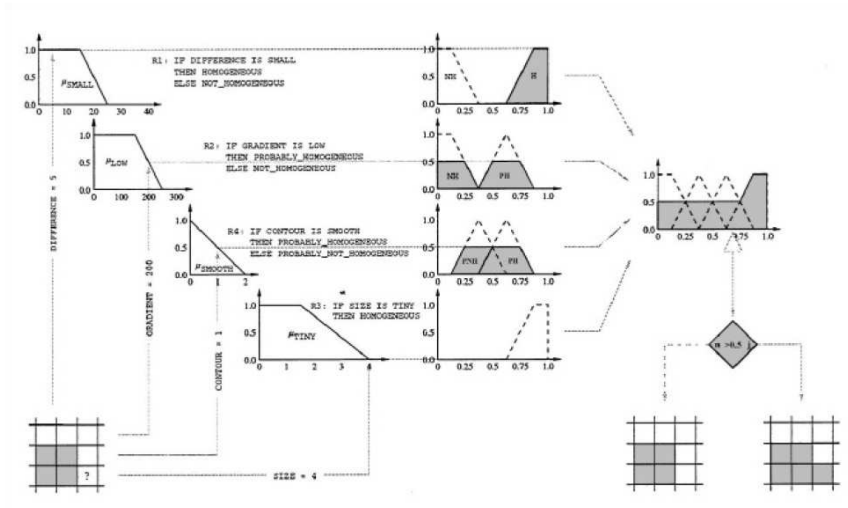


Fig. 7. An example for the evaluation of the fuzzy rule-based homogeneity criterion.